

SOUND PROPAGATION IN DUCTS
WITH
ABSORBENT LININGS

EKSAMENSARBEJDE

I

LYDTRANSMISSION I ABSORPTIONSBEKLÆDT RØR

M.Sc. THESIS

by

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FOREWORD

Many noise problems can be traced to sound transmitted in pipes and ducts; this is the case for example with noise from ventilation systems, chimneys/funnels or motor exhaust systems. Mufflers are used in order to limit this noise; among those widely used are absorptive mufflers, called silencers. It is for example usual to employ silencers in ventilation ducts. To reduce the noise from motor exhaust systems a combination of reflective mufflers and silencers is often used.

The aim of this thesis is, on the basis of a state of the art study, to develop an algorithm to describe sound propagation in ducts with absorbent linings. It has been necessary to employ iterative procedures including Newton-Raphson's method.

Calculations using the algorithm have been compared with both published calculations and measurements. The algorithm has also been tested with measurements made by the author on a silencer generously donated by Ødegaard & Danneskiold-Samsøe A/S.

Many thanks are due, for the active support and encouragement given me by my supervisor Finn Jacobsen.

FORORD

Mange støjproblemer kan henføres til lyd transmitteret i rør eller kanaler; dette gælder f.eks. støj fra ventilations-anlæg, skorstene eller motorudstødningssystemer. For at begrænse denne støj anvendes lyddæmpere; bl.a. anvendes i vid udstrækning lyddæmpere, som indeholder absorberende materiale. F.eks. er det almindeligt at anvende absorptionslyddæmpere i ventilationskanaler. Til dæmpning af støjen fra motorudstødningssystemer anvendes ofte en kombination af reflektions- og absorptionslyddæmpere.

Formålet med projektet er, ud fra en litteraturgennemgang, at opstille en beregningsmodel (algoritme) for lydudbredelse i et absorptionsbeklædt rør. Det viste sig at være nødvendigt at anvende iterative numeriske metoder til løsning af problemet, bl.a. Newton-Raphsons metode.

Den udviklede algoritme er, i en vis udstrækning, blevet afprøvet ved sammenligning med foreliggende måleresultater. Algoritmen er også blevet afprøvet med målinger foretaget på en lyddæmper givet af Ødegaard & Danneskiold-Samsøe A/S.

Jeg er meget taknemmelig for det aktive støtte og hjælp min vejleder Finn Jacobsen har givet mig.

This thesis is dedicated to my wife and children

Susanne, Martin and Maria-Therese

Michael Howard Grant

1. INTRODUCTION

For loss free sound transmission in hard walled ducts of uniform cross-section the associated propagation constant is imaginary for those modes above their cut on frequency. When losses are taken into consideration the propagation constant associated with the wave becomes complex, the real part of which represents the attenuation. For those modes below their cut on frequency the associated propagation constant is always completely real and the wave is evanescent. The presence of absorption in this latter case is just to increase the attenuation by adding a real number to the propagation constant. Interestingly the transmission of sound in ducts of varying cross-section is akin to the phenomenon of barrier penetration in quantum mechanics, see [ref. 1]. Evanescent waves can penetrate narrow constrictions in ducts in a similar fashion to the penetration of particles through a potential barrier. In this latter case it is a question of an evanescent solution to Schrödinger's wave equation, within the barrier.

The models below that attempt to describe sound absorption have as their aim the determination of the complex propagation constant.

The problem of sound propagation in ducts with absorptive linings was dealt with by Morse in 1939 [ref. 2] where he considered the problem from the point of view of locally reacting linings, this is where the effect of the lining on the passage of sound can be wholly described in terms of its acoustic impedance, Scott, [ref. 3; § 6]. Such linings exist where the viscous forces are high or where the impedance is much greater than that of air. The angle of incidence from air into the lining is very nearly zero. Local reaction also occurs where the lining is separated by closely spaced partitions or baffles that inhibit wave propagation along the duct lining, see [ref. 1] and Fig. 1.1 below.

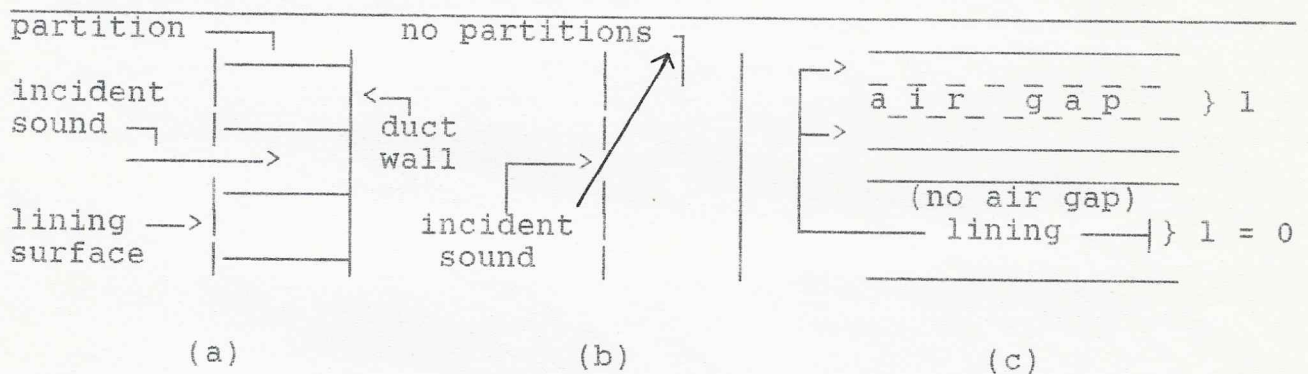


Fig. 1.1. (a) Locally reacting linings. (b) Non locally reacting linings. (c) The assumption that the duct liner in (b) behaves like the liner in (a) leads to the incorrect conclusion that the attenuation becomes infinite as the width of the air gap, l , goes to zero.

A simple demonstration as indicated in Fig. 1.1 (c) shows that the assumption of local reaction for liners leads to the incorrect

conclusion that the attenuation becomes infinite as the width of the air gap, l , goes to zero.

The propagation constant for the duct and lining should be the same in this case. To fulfil this requirement one must take account of the bulk or extended reaction of the lining material.

Scott in a paper published in 1946 [ref. 3] seems to have been the first to develop a theory for extended (or bulk) reaction. In this case there is wave propagation within the duct lining parallel to the duct walls/axis. He suggested that the equations describing wave propagation in both the lining and the air space can be represented by a single equation containing a common propagation constant for the whole duct. His analysis assumed homogeneous and isotropic linings, Scott, [ref. 4], ("...in so far as concerns the propagation of acoustic waves.") [ref. 3; § 3] and the cross-section of the duct being either rectangular or circular.

The equations he derived are transcendental as is also the case for Morse's theory and they can only be solved using iterative methods. Christie [ref. 5] in a paper published in 1971 solved for a range of flow resistivities both the Morse and Scott equations using a Newton-Raphson method. In Christie's paper the lining propagation constant and characteristic impedance as functions of frequency were derived from a single value of the flow resistivity using the empirical equations of Delany and Bazley, [refs. 6 & 7]. Christie compared results for both local and extended reaction which showed large discrepancies for low flow resistivities, but good agreement when the wavelength is either much greater than or much less than the duct width. See Fig. 1.2 on p. 4 where comparison is also made with calculations made by the author. The other physical parameters of the duct such as lining thickness and width of air space were kept constant. Please note that Christie uses the term flow resistance giving it the dimensions of rayl/m that is $N s m^{-4}$, these are in the fact the dimensions of flow resistivity.

Building on the work of Christie the present thesis includes an algorithm, 'Extended', for solving extended reaction. This has been used in conjunction with the four-pole programme devised by Finn Jacobsen which uses the four-pole approach to duct acoustics described in [refs. 7 & 8]. The algorithm 'Extended' fulfils the requirement that as the air gap, l , goes to zero the propagation constants for the duct and lining are the same. Some results using both 'Extended' and the approximate local reaction calculations with an earlier programme ('Four pole') are also shown in Fig. 1.2 for comparison with [ref. 5].

In the course of investigations both the flow resistivity and the other parameters such as lining thickness were varied. It is of course the ratio b of lining thickness to duct width that is important (see [refs. 5 & 9]) although the absolute value of the duct width will determine the cut on frequencies for the higher order modes.

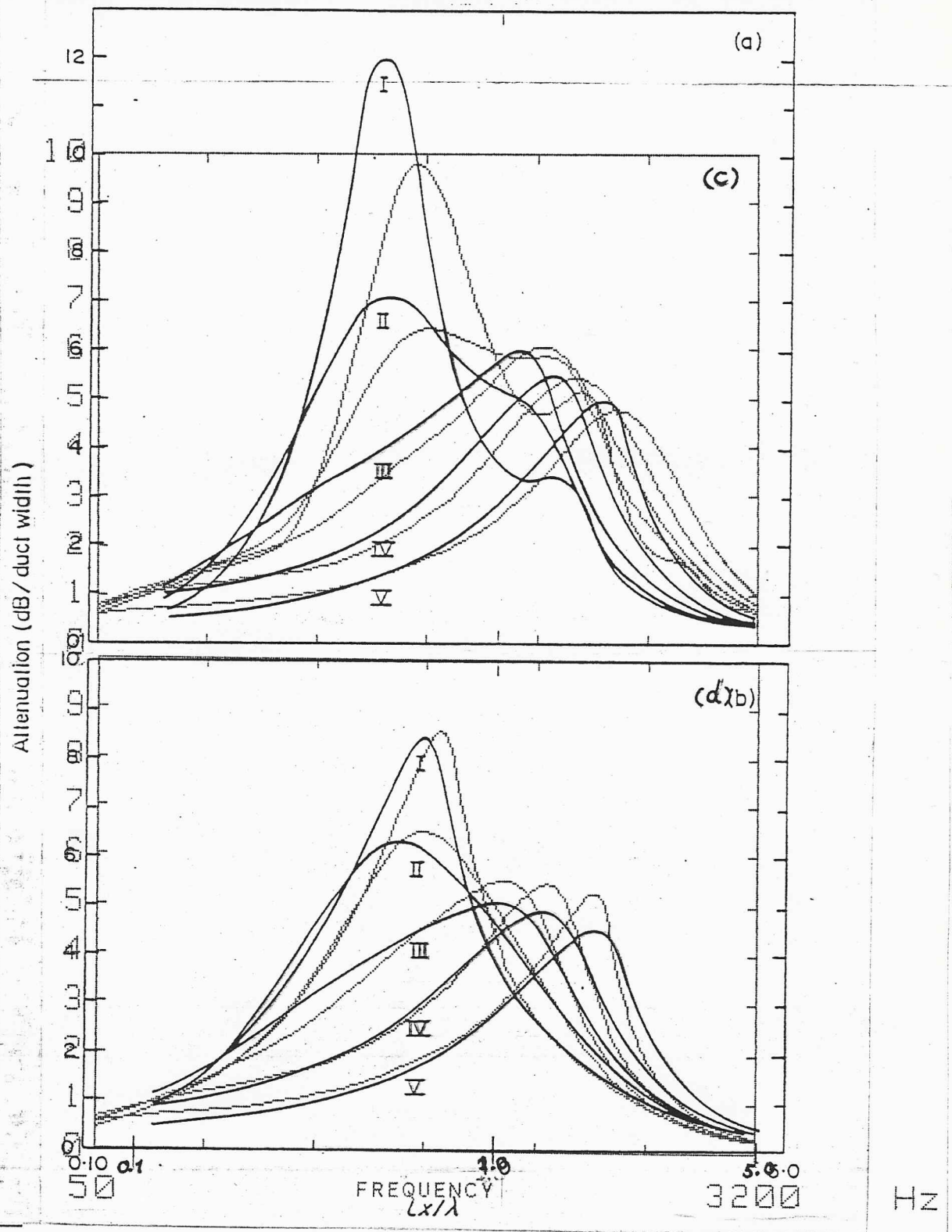


Fig. 1.2. The transparency is taken from Christie [ref. 5].
 (a) Morse's analysis see eq. 2.2.17
 (b) Scott's analysis see eq. 2.2.9
 (c) Morse's analysis using the 'Four_pole' programme
 (corrected for rectangular ducts using eq. 2.1.3)
 (d) Scott's analysis using the algorithm 'Extended'

Duct width, $l = 0.5334$ m; lining thickness $d = 0.1524$ m

	I	II	III	IV	V
Flow Resistivity	4000	8000	16000	32000	64000

At these cut on frequencies experience has shown that convergence problems can arise with the iteration; one may start to follow a series of solutions that belong to another mode of propagation than that with which the iteration began. Around cut on, the $(0,2)$ mode can sometimes begin to be less attenuated than the plane wave, $(0,0)$, mode or have an attenuation that is very close to the attenuation of the plane wave, see for example § 3.3. and [ref. 9; Fig. 4a.].

It is dubious as to whether it is really correct to talk in terms of the $(0,0)$ or $(0,2)$ mode etc. since these have only any real meaning for hard walled ducts.

During the course of the thesis the author has found it useful to keep in mind the following mental picture to use as an aid in understanding the propagation of sound in ducts.

Imagine a piston sound source radiating into a straight duct of circular cross-section, with an anechoic termination and with perfectly absorbing walls. The piston would radiate in the same way as if it were placed flush with an infinite baffle and radiated into the 'half space' as mentioned in § 4.1 of this thesis. For low frequencies the angle of incidence into the lining near the piston would be very nearly zero because of the spherical directivity pattern. Applying Huygen's principle, though somewhat loosely, allows this argument to be applied to any arbitrary cross-section along the length of the duct. There would be a situation that could quite well be described by Morse's theory. The sound would be constantly being diffracted out towards the sides of the duct. In fact the idea of plane wave propagation can be seen to already break down because of the presence of the lining. At this point it is informative to quote from Morse and Ingard [ref. 10; p.509.]

"...it becomes apparent that very few sweeping generalities can be made concerning the behaviour of sound in ducts with highly absorbent walls..." and further on they write "... the principal wave is far from a uniform plane wave, the negative reactance of the walls having in some way pulled most of the energy of the wave away from the centre of the duct to the periphery, where it is more quickly absorbed as it travels along."

As the frequency is increased the directivity pattern becomes less spherical and the sound radiates out more as a diverging beam. The sound will be incident on the lining with an increasing angle of incidence. Let us now replace the lining by one that is less perfectly absorbing and which permits sound to propagate within its bulk. Some of the sound will propagate within the lining parallel to the duct axis but there will also be grazing incidence whereupon that portion of the sound will be completely reflected. This follows from Rayleigh's reflection coefficient ([ref. 11]). If around the lining surface there is a perforated metal cylinder then more sound will be reflected with an accompanying decrease in attenuation. This is the opposite of what one may expect with local reaction, see the discussion in § 4.1 where there is also a description of how this hypothesis has been tested. When the

frequency is increased still further the beaming effect becomes even more marked and plane progressive waves travel down the duct parallel to both the axis and the surface of the lining material. Very little sound is propagated within the bulk of the lining, the sound is transmitted unhindered down the duct and so the attenuation falls to zero. That this is so is borne out in Fig. 1.2. Somewhere between the extremes mentioned above there will be peak attenuation for that particular combination of duct geometry and lining.

Morse's theory also predicts peak(s) of attenuation but for different reasons. In the case of locally reacting linings the peaks correspond to those frequencies given by $2(m + 1)c/(4d)$, where d is the lining thickness and $m = 0, 1, 2, \dots$ etc. The phenomenon is physically different to that described above as is borne out by Christie's paper wherein it can be seen that the attenuation predicted by Scott does rise from zero to a maximum and fall again to zero but the maxima for Scott's model and those for Morse's model do not always correspond to the same frequency. See again Fig. 1.2.

Scott's theory is outlined in § 2 together with some comments on the use of Delany and Bazley's empirical formulae. The work of Wassilieff [ref. 9] is also commented upon.

The algorithm 'Extended' is described in § 3 with a discussion about its construction and use especially concerning the two different iterative methods employed. Results of some further calculations are shown and discussed.

Measurements on a silencer to determine the insertion loss (IL) are described in § 4. A description is also given of the measurement of flow resistance for a sample of material taken from the silencer lining. Calculations of the predicted IL using the algorithm 'Extended' are compared with experimental results and discussed.

The conclusion in § 5 reviews the thesis and has suggestions for further investigations.

2. THEORY

2.1. General Considerations

The theory used throughout this thesis is that developed by Scott in [ref. 3; § 3] in which it is assumed that the duct is rectangular. Scott further assumes that by symmetry the component of the particle velocity normal to the central plane of the duct is zero. This boundary condition limits the analysis to those modes that have a plane of pressure amplitude maxima along the duct axis.

Wassilieff [ref. 9; § 2] has rightly interpreted this but has unfortunately written that there "...is zero particle velocity at the centre line of the duct axis." and that only the symmetric modes (0,0), (0,2) and (0,4) etc. can give solutions to Scott's equations. This is in fact incorrect for the (0,0) mode which cannot always give zero particle velocity along the centre line. What should be written is that the component of the particle velocity normal to the central plane parallel to the absorbing sides is zero. 'Kinsler & Frey' [ref. 11; § 9.8; Fig. 9.8] and Morse and Ingard [ref. 10; § 9.2; pp.493-494] show schematically that for the (0,1) mode there is a nodal plane for pressure amplitude midway between the walls of a rectangular, hard walled, duct; this is a plane of maximum normal particle velocity amplitude a condition at odds with that required in Scott's paper. The possible modes of propagation in a rectangular duct are (m,n) where m or n is 0 or any positive integer, see again [refs. 10 & 11]. The (m,0) mode is absorbed by the wall linings similarly for all values of m in Scott's analysis since the absorbing material is placed symmetrically about the central plane, which is coplanar with $y = 0$, see Fig. 2.1. The analysis is as valid for an air space between two parallel absorbing planes of infinite extent as for a rectangular duct with two parallel walls absorbing and the others totally reflecting. The modes of propagation that can therefore be distinguished as having different propagation constants are (0,n) where $n = 0, 2, 4$, etc. as in [ref. 9].

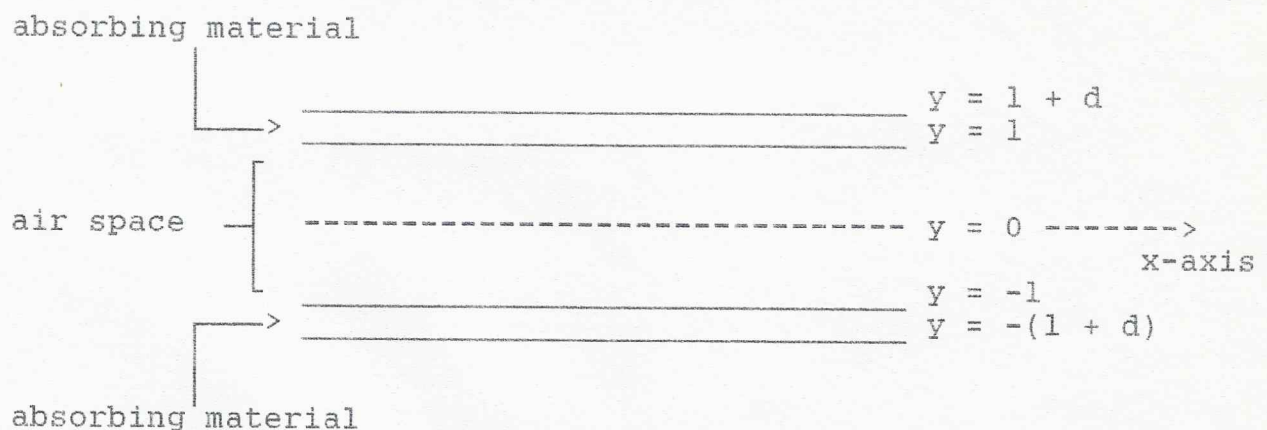


Fig. 2.1. Plan of channel as shown in [ref. 3].

The properties of the porous material are regarded as being solely described by its flow resistivity and the frequency as shown by Delany & Bazley [ref. 6] and as modified by Mechel, below, see [ref. 12], see also Vér [ref. 13; eqs. 14 & 15] and Munjal [ref. 14; eqs. 6.29 & 6.30].

For $A \leq 60$; where $A = \sigma/\rho \cdot f$

$$\Gamma_a/k = 0.189 \cdot A^{0.6185} + j \cdot \left[1 + 0.0978 \cdot A^{0.6929} \right] \quad 2.1.1$$

$$Z_a/Z_o = 1 + 0.04885 \cdot A^{0.754} - j \cdot 0.087 \cdot A^{0.7307}$$

For $A \geq 60$

$$\Gamma_a/k = \left[-1.4664 + j \cdot 0.2115 A \right]^{\frac{1}{2}} \quad 2.1.2$$

$$Z_a/Z = \frac{(A/2\pi) + j \cdot 1.4035}{\Gamma_a/k}$$

Please note that there is a printing error in Mechel's original article and mistakes of a more gross nature have unfortunately found their way into [refs. 13 & 14] as referred to above. The equations which are also to be found correctly printed in [ref. 12] are as written above. These form the basis of the subroutine 'Material' used together with 'Extended' in the four-pole programme.

Scott has also derived an equation for the propagation constant for ducts with circular cross sections lined with a layer of uniform thickness of homogeneous, isotropic porous material, see [ref. 3; §4]. This is again transcendental and contains a number of Bessel and Neumann functions of zeroth and first order, the solution of which is not attempted in this thesis. Mechel in [ref. 15] describes an iterative procedure for "...absorbers with sound propagation parallel to the duct axis in the interior of the absorber material,..." and gives in an appendix the numerical computation of Bessel functions.

This thesis deals with ducts of circular or non-rectangular cross-sections and rectangular ducts with absorbing material on all sides, in the manner described by Mechel in [ref. 16] and which was arrived at independently by the author using the following argument.

From Munjal [ref. 14; § 6.6; p. 244] the acoustic attenuation is known to be proportional to the ratio of lined perimeter to flow area. The attenuation is directly proportional to the real part of the propagation constant ($\text{Re}\{\Gamma\}$). Let R be the radius and H and W the height and width of circular and rectangular ducts respectively, then the real part of the propagation constant can be written as follows:

$\text{Re}\{\Gamma(p)\}$	$\frac{2 \cdot H}{H \cdot W}$	for rectangular ducts with porous material on two parallel sides
$\text{Re}\{\Gamma(s)\}$	$\frac{2 \cdot (H + W)}{H \cdot W}$	for rectangular ducts with porous material all around the inside
$\text{Re}\{\Gamma(c)\}$	$\frac{2 \cdot \pi \cdot R}{\pi \cdot R \cdot R}$	for circular ducts with porous material all around the inside

If the rectangular duct is square then $W = H$ and if $2 \cdot R = H$ then one has:

$$\text{Re}\{\Gamma(p)\} = 2 \cdot \text{Re}\{\Gamma(s)\} = 2 \cdot \text{Re}\{\Gamma(c)\} \quad 2.1.3$$

This factor of 2 is accounted for in the subroutine and algorithm 'Extended' in programme lines 10960 to 11060. The approach adopted above is assumed to be valid for plane wave propagation but not for higher order modes where the actual duct geometry plays a more important role. It is further assumed that the lining thickness is constant and that there is no flow present, the Mach number is therefore zero. Viscous losses at the walls are ignored and the medium (air) is also assumed to be inviscid.

2.1.1. Some Comments on the Four-Pole Method

It is interesting to note that the four-pole matrix, see [ref. 7; § 4] is mathematically identical to the Mach or Lorentz transformation. This is actually not too surprising since the whole idea behind the Mach transformation is to leave the wave equation invariant. The wave maintains its shape and phase velocity leading to what is called the invariance of the interval. The wave has this behaviour also in the mathematically banal case of a lateral translation, which is after all what is happening in the four-pole example where the point of observation is translated from one end of the duct to the other. The Mach transformation can be put to use in duct acoustics when flow is present; see the hand written example in the Appendix.

2.2. Scott's Theory

The field equations for the air space and lining respectively are:

$$\begin{aligned} v &= -\text{grad } \bar{\Phi}, \quad p = j \cdot t \cdot \bar{\Phi}, \quad \nabla^2 \bar{\Phi} + k^2 \bar{\Phi} = 0 \\ v &= -\text{grad } \theta, \quad p = j \cdot t \cdot \theta, \quad \nabla^2 \theta + h^2 \theta = 0 \end{aligned} \quad 2.2.1$$

$$\begin{aligned} \operatorname{Re}\{\Gamma(p)\} &\propto \frac{2 \cdot H}{H \cdot W} && \text{for rectangular ducts with porous material on two parallel sides} \\ \operatorname{Re}\{\Gamma(s)\} &\propto \frac{2 \cdot (H + W)}{H \cdot W} && \text{for rectangular ducts with porous material all around the inside} \\ \operatorname{Re}\{\Gamma(c)\} &\propto \frac{2 \cdot \pi \cdot R}{\pi \cdot R \cdot R} && \text{for circular ducts with porous material all around the inside} \end{aligned}$$

If the rectangular duct is square then $W = H$ and if $2 \cdot R = H$ then one has:

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2.2. Scott's Theory

The field equations for the air space and lining respectively are:

$$\begin{aligned} v &= -\operatorname{grad} \Phi, \quad p = j \cdot \omega t \cdot \nabla^2 \Phi + k^2 \Phi = 0 \\ v &= -\operatorname{grad} \theta, \quad p = j \cdot \omega t \cdot \nabla^2 \theta + h^2 \theta = 0 \end{aligned} \quad 2.2.1$$

Scott suggested that for any one mode of propagation along the duct Φ and Θ will have a common propagation constant Γ such that for a rectangular duct, see Fig. 2.1:

$$\begin{aligned}\Phi &= \Phi(y) \cdot e^{-\Gamma x} \cdot e^{j\omega t} \\ \Theta &= \Theta(y) \cdot e^{-\Gamma x} \cdot e^{j\omega t}\end{aligned}\tag{2.2.2}$$

Substituting these equations into the field equations above for the air space and lining respectively gives:

$$\begin{aligned}\Phi(y) &= A \cdot e^{j\lambda y} + B \cdot e^{-j\lambda y} ; \quad \lambda^2 = \Gamma^2 + k^2 \\ \Theta(y) &= C \cdot e^{j\mu y} + D \cdot e^{-j\mu y} ; \quad \mu^2 = \Gamma^2 + h^2\end{aligned}\tag{2.2.3}$$

Where A, B, C and D are constants yet to be found.

Scott argues that along the central plane $y = 0$ the gradient $\partial\Phi/\partial y = 0$ by symmetry; this is the condition for zero normal particle velocity (see § 2.1). Therefore

$$A = B$$

At the outer wall (assumed rigid) $y = l+d$ and $\partial\Theta/\partial y = 0$ i.e. the particle velocity normal to the wall is zero therefore

$$j\mu C \cdot e^{j\mu(1+d)} - j\mu D \cdot e^{-j\mu(1+d)} = 0$$

or

$$D = C \cdot e^{2j\mu(1+d)}\tag{2.2.4}$$

(N.B. in Scott's original paper d is the lining thickness and l is the width of the air space).

At the lining/air surface, $y = l$, there is both continuity of normal particle velocity and acoustic pressure giving

$$j\lambda A \cdot e^{j\lambda l} - j\lambda B \cdot e^{-j\lambda l} = j\mu C \cdot e^{j\mu l} - j\mu D \cdot e^{-j\mu l}$$

and

$$\rho \cdot [A \cdot e^{j\lambda l} + B \cdot e^{-j\lambda l}] = \rho' \cdot [C \cdot e^{j\mu l} + D \cdot e^{-j\mu l}]\tag{2.2.5}$$

Substituting for B and D and eliminating A and C gives:

$$\varrho \cdot \frac{\coth(-j\lambda l)}{\lambda} = \varrho' \cdot \frac{\coth(j\mu d)}{\mu} \quad 2.2.6$$

The above equation can be written in the following way

$$\mu \varrho \cdot \tan(\mu d) = -\lambda \varrho' \cdot \tan(\lambda l)$$

which using eq. 2.2.3 can be written as

$$\varrho \cdot \left[\Gamma^2 + h^2 \right]^{\frac{1}{2}} \tan \left[\left[\Gamma^2 + h^2 \right]^{\frac{1}{2}} d \right] = -\varrho' \cdot \left[\Gamma^2 + k^2 \right]^{\frac{1}{2}} \tan \left[\left[\Gamma^2 + k^2 \right]^{\frac{1}{2}} l \right] \quad 2.2.7$$

Christie [ref. 5] uses the following variables:

$$w = \left[(\Gamma l)^2 + (k l)^2 \right]^{\frac{1}{2}}; \quad p = \left[(h l)^2 - (k l)^2 \right]^{\frac{1}{2}}; \quad b = d/l; \quad M = \varrho / \varrho' \quad 2.2.8$$

allowing eq. 2.2.7 to be written in the following manner:

$$f(w) = w \cdot \tan w + M \left[w^2 + p^2 \right]^{\frac{1}{2}} \cdot \tan \left[b \left[w^2 + p^2 \right]^{\frac{1}{2}} \right] = 0 \quad 2.2.9$$

This equation is as mentioned before transcendental and has therefore to be solved either using the Newton-Raphson method or some other iterative procedure such as Muller's method, see [ref. 17; § 2.5].

2.2.1. The properties of the lining

Christie [ref. 5] leaves the factor M as the ratio ϱ / ϱ' stating that eq. 2.2.9 is solved with the help of Delany and Bazley's empirical formulae, without showing how M has been expanded and with which assumptions. (Writing before the modifications of Mechel, these formulae must necessarily be the original version in [ref. 6] which may in part account for the slight discrepancies between Christie's calculations and those obtained by the author, see Fig. 1.1.)

Kurze and Vér [ref. 18; § 2] writing, about non-isotropic linings, a year later than Christie, argue that: "Compression losses are small and the bulk modulus K is assumed to be real. At low frequencies, the compression in fibrous materials is isothermal and K equals the ambient static pressure P. Only at high frequencies does K assume the free-field value for adiabatic compression, i.e. $K = 1.4P = \varrho \cdot c^2$, where c is the free-field sound velocity".

Wassilieff [ref. 9; § 2] takes exception to this, writing concerning [ref. 18], that: "There is a factor $\varrho \cdot c^2 / K$ lacking, where K is the bulk modulus of the lining. This has probably

occurred as a result of the common practice of assuming it to be equal to unity. The magnitude of $\rho \cdot c^2 / K$, in fact, varies from 1.4 at low frequencies to 1.0 at high frequencies. This approximation renders the predictions of Kurze and Vér's equation conservative, thus providing a possible explanation for their own observations regarding poor agreement with experiment". See also the comments on eq. 2.2.20 in § 2.2.2 on p. 19.

With the aid of eq. 2.2.1, the boundary condition of equal normal particle velocity at the lining/air interface which in [ref. 9; § 2] is written as

$$\partial \Phi / \partial y = \partial \Theta / \partial y$$

can be written as follows by rewriting Φ and Θ as

$$\Phi = (1/j\omega\rho)p(\text{air}) \text{ and } \Theta = (1/j\omega\rho')p(\text{lining})$$

and writing $p(\text{air})$ as p and $p(\text{lining})$ as p' , thus giving

$$\partial p / \partial y = (\rho / \rho') \partial p' / \partial y \quad 2.2.10$$

Scott in [ref. 4; § 2; eq. 8] shows that the complex density ρ' can be written (using Scott's notation) as:

$$\rho' = \hbar^2 \kappa / P \omega^2 \quad 2.2.11$$

where (κ) κ is called by Scott the effective compressibility of the air in the pore space and P , in this case, is the porosity. Scott writes: "...by definition $p = s\kappa$..." where s is the condensation, it can be seen therefore that κ is in fact the bulk modulus K , see [ref. 14; § 5.2; eq. 5.5; p. 101]. (Convention today has that compressibility and bulk modulus are mutually reciprocal.)

The complex density is related to the characteristic impedance of the lining in the following way, (see Scott [ref. 4; § 3; eq. 13])

$$Z_a = \rho' \omega / \hbar \quad 2.2.12$$

Which on substitution into eq. 2.2.10, and writing $\omega = ck$, gives

$$\partial p / \partial y = (\rho ck / Z_a \hbar) \partial p' / \partial y \quad 2.2.13$$

Also, substituting for ρ' in eq. 2.2.10 with eq. 2.2.11 (with $\omega = ck$ and $\kappa = K$) gives

$$\partial p / \partial y = (\rho c^2 / K) \cdot (k^2 P / \hbar^2) \partial p' / \partial y \quad 2.2.14$$

Identifying the porosity P , above, with σ in [refs. 9 & 18] and noting that

$$j\hbar = jka_i [\text{ref. 18; eq. (4)}] = \gamma_l = \alpha + j\beta [\text{ref. 9; eq. (1)}]$$

then eq. 2.2.14 above, as shown by Wassilieff [ref. 9], reveals the factor of $\rho c^2 / K$ missing from eq. (12) Kurze and Vér [ref. 18].

Further, comparison of the above analysis with that of [ref. 9] shows that ρ/ρ' in [ref. 5] can be written as $\rho \cdot ck/Z_a h$ which on writing

$$\Gamma_a = jh = \alpha + j\beta;$$

gives

$$M = -j\rho \cdot ck/Z_a \Gamma_a$$

which can be identified with line 9230 in the algorithm 'Extended'. Here the value of ρ/ρ' is obtained directly from the values of Γ_a and Z_a derived from the Delany and Bazley formulae eqs. 2.1.1 and 2.2.2 as written into the subroutine 'Material'.

2.2.2. Low and high frequency values of w

Wassilieff [ref. 9; § 3] examines both the starting and finishing points for w in the complex plane ($\text{Re}\{w\}, \text{Im}\{w\}$) arguing as follows "...for bulk reacting linings the starting point for the (0,2) mode can be less than $\pi + j \cdot 0$. To demonstrate this behaviour... For linings of low flow resistivity ... $\rho' \rightarrow \rho$. Under these conditions equation... can be written as $w \cdot \tan w \approx -\tan[(d/l) \cdot w]$. Solutions to this equation are given by

$$w = [n\pi/(1 + (d/l))] + j \cdot 0 \text{ as } k \rightarrow 0$$

where $n = 0$ for the (0,0) mode and $n = 1$ for the (0,2) mode. The physical interpretation of these solutions is that the duct "looks wider" by a factor $1 + (d/l)$ at low frequencies and low lining flow resistivities, as would be intuitively obvious." Wassilieff illustrates these ideas in Fig. 3 of the above reference (reproduced in this thesis as Fig. 2.2) where the low frequency (20 Hz) starting point is shown very close to $\frac{1}{2}\pi + j \cdot 0$.

The author has found that calculations with 'Extended' together with 'Material' do not for the (0,2) mode give results in agreement with the above, but rather that the starting value for w is as for local reacting linings, $\pi + j \cdot 0$. It is possible though that Wassilieff has modified Delany and Bazley's formula or used an iteration where the flow resistance varies with frequency. This is not the impression the author has received on reading [ref. 9]. Figs. 2.3 and 2.4 show the results of calculations with 'Extended', for comparison with Fig. 2.2. Fig. 2.4 shows a more detailed investigation of the results in Fig. 2.3.

The Nyquist plots are provided without a scale in all figures where they appear, apart from the reproduction in Fig. 2.2. Their function is purely qualitative and provided the author with an immediate overall view as to how the iteration progressed. Sudden deviations to solutions belonging to another mode of propagation would show up as 'jumps' in the complex plane, such as indicted by the arrows in Fig. 2.2. The value of w at each frequency considered was printed in an array when a more detailed analysis was desired. For legibility, there are no tables in this report, the author has sufficed to comment on the appropriate details.

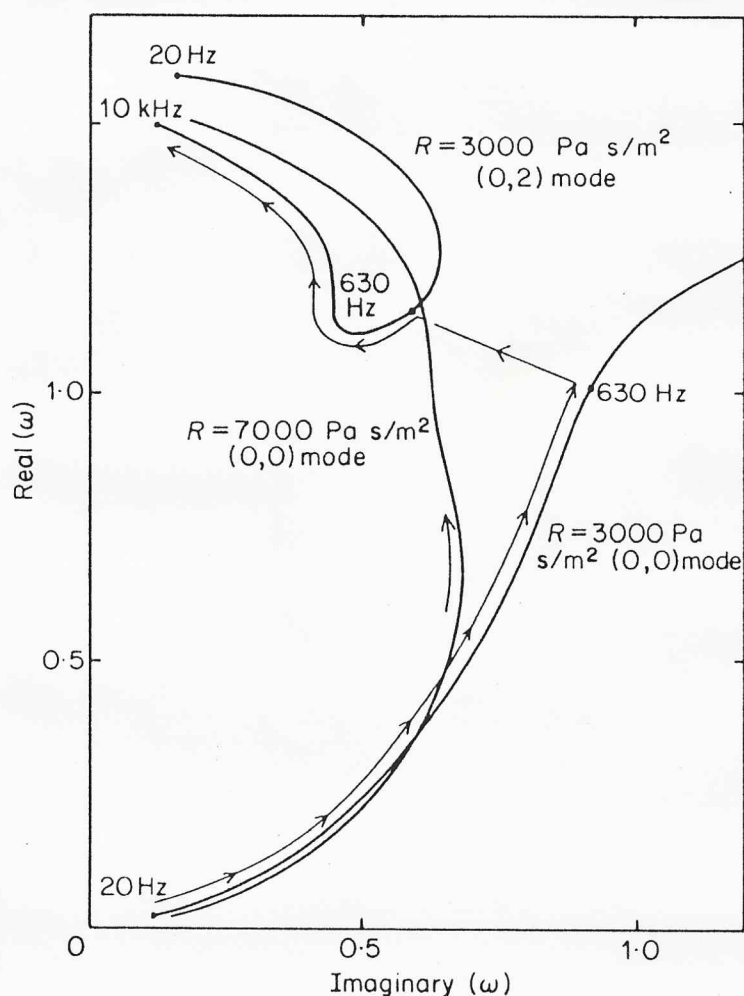


Fig. 2.2. Reproduced from [ref. 9]. Plot of $\text{Re}\{w\}$ versus $\text{Im}\{w\}$ calculated from eq. (7), [ref. 9] (in the isotropic case here, this is identical with eq. 2.2.9). Duct width $l = 0.3$ m lining thickness $d = 0.15$ m, flow resistivities (R in fig. 3) $\sigma = 3000$ N·s/m⁴ and 7000 N·s/m⁴. According to Wassilieff the arrows follow the roots yielding least attenuation.

The author has found that the high frequency end point for the least attenuated mode (either (0,0) or (0,2)) is $\frac{1}{2}\pi + j \cdot 0$, this is in agreement with Wassilieff [ref. 9] and plate IV in [ref. 10]. Furthermore, calculations with 'Extended' show that the (0,2) mode can be the least attenuated mode. This is borne out by Wassilieff, in Fig. 3 (Fig. 2.2 in this thesis) and Fig. 4a. of [ref. 9].

Concerning low frequency values of w , Wassilieff's argument holds if Γ_a and k approach zero at the same rate and in such a way that their ratio approaches $0 + j$, see eq. 2.1.1. It is shown below that this is not the case and that in the low frequency limit M goes to zero (i.e. φ' does not approach φ) and that $\pi + j \cdot 0$ is the starting point for w for the (0,2) mode as is also the case with Morse's equation for locally reacting linings.

An examination of Delany and Bazley's formulae as given in § 2,

eqs. 2.1.1 and 2.1.2 reveals the following:

$A = \sigma/\rho \cdot f$ where σ is constant as is assumed in [ref. 9] and $k = 2\pi f/c$, therefore,

$$\lim_{f \rightarrow 0} A = \infty; \quad \lim_{f \rightarrow 0} k = 0; \quad \lim_{f \rightarrow 0} A \cdot k = 2\pi\sigma/\rho \cdot c; \quad \lim_{f \rightarrow 0} A \cdot k^2 = 0.$$

From eq. 2.1.2 (for $A \geq 60$) the following may be written

$$\Gamma_a = \sqrt{[-1.4664 k^2 + j \cdot 0.2115 A \cdot k^2]} \quad 2.2.15$$

and replacing Z_o by $\rho \cdot c$

$$(Z_a/\rho c)(\Gamma_a/k) = [(A/2\pi) + j \cdot 1.4035] \quad 2.2.16$$

therefore $\lim_{f \rightarrow 0} \Gamma_a = 0$ as in [ref. 9] but $\lim_{f \rightarrow 0} (Z_a/\rho \cdot c) \cdot (\Gamma_a/k) = \infty$.

This latter is because $k \rightarrow 0$ for decreasing frequency faster than does Γ_a . Now, $(Z_a/\rho c) \cdot (\Gamma_a/k) = +j/M$ and so as $f \rightarrow 0$, $M \rightarrow 0$ and Scott's equation equals Morse's equation (see below) at low enough frequencies. Christie [ref. 5] has shown that Morse's equation can be written as

$$w \cdot \tan w = -jkl/Z_a. \quad 2.2.17$$

(It should of course be borne in mind that in the low frequency limit (as $k \rightarrow 0$) w approaches Γ_l , see eq. 2.2.8.) Comparison with the equations at the bottom of p. 508 [ref. 10] and using eq.(9.2.20) together with plate IV, all in the same reference, reveals that solutions to the (0,0) mode start at $0 + j \cdot 0$ and solutions for the (0,2) mode start at $\pi + j \cdot 0$. By the aforementioned reasoning this is necessarily the same for Scott's equation in the low frequency limit. From eq. 2.2.13 above one can write;

$$\lim_{f \rightarrow 0} Z_a/\rho \cdot c = (A/2\pi)/(\Gamma_a/k) = \sigma/(\Gamma_a \rho \cdot c) \quad 2.2.18$$

This increases with decreasing frequency which is at variance with Wassilieff [ref. 9; § 3] where it is written that $R \rightarrow 0$ as $k \rightarrow 0$ (where R equals $\text{Re}\{Z_a\}$). For a constant flow resistivity k can be chosen arbitrarily small so that Z (both $\text{Re}\{Z_a\}$ and $\text{Im}\{Z_a\}$) can be made arbitrarily large. R will only approach zero if σ approaches zero faster than does Γ_a . In this case $A \rightarrow 0$ and one must use eq. 2.1.1, and $\Gamma_a/k \rightarrow 0 + j$ as mentioned previously. This is obviously not the case for constant σ as is assumed in both [ref. 9] and in the theory used throughout this thesis. Measurements of the flow resistivity (see § 4.2) of material taken from a silencer do show a sudden drop towards zero in the value of $\text{Re}\{\sigma\}$ under 8 Hz. The sound source used in this measurement is a small loud speaker and it is highly questionable whether this is a reliable result given, at these frequencies, both the poor radiation factor for loudspeakers and the high pass characteristic of the filter employed.

The following simple numerical example taken from Fig. 2.2 (Fig. 3 in [ref. 9]) serves to highlight the above.

From Fig. 2.2 where $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}^4$ Wassilieff has, for the (0,2) mode, w starting (when $f = 20 \text{ Hz}$) near to $\frac{1}{2}\pi + j\cdot 0$ for the reasons given at the beginning of § 2.2.2. However these values for σ and f give with the density of air $\rho = 1.205 \text{ kg}/\text{m}^3$ a value $A = 124.451$. One has therefore to use eq. 2.1.2 ($A \geq 60$), assuming validity at this frequency. With $c = 344.246 \text{ m/s}$ one obtains the following:

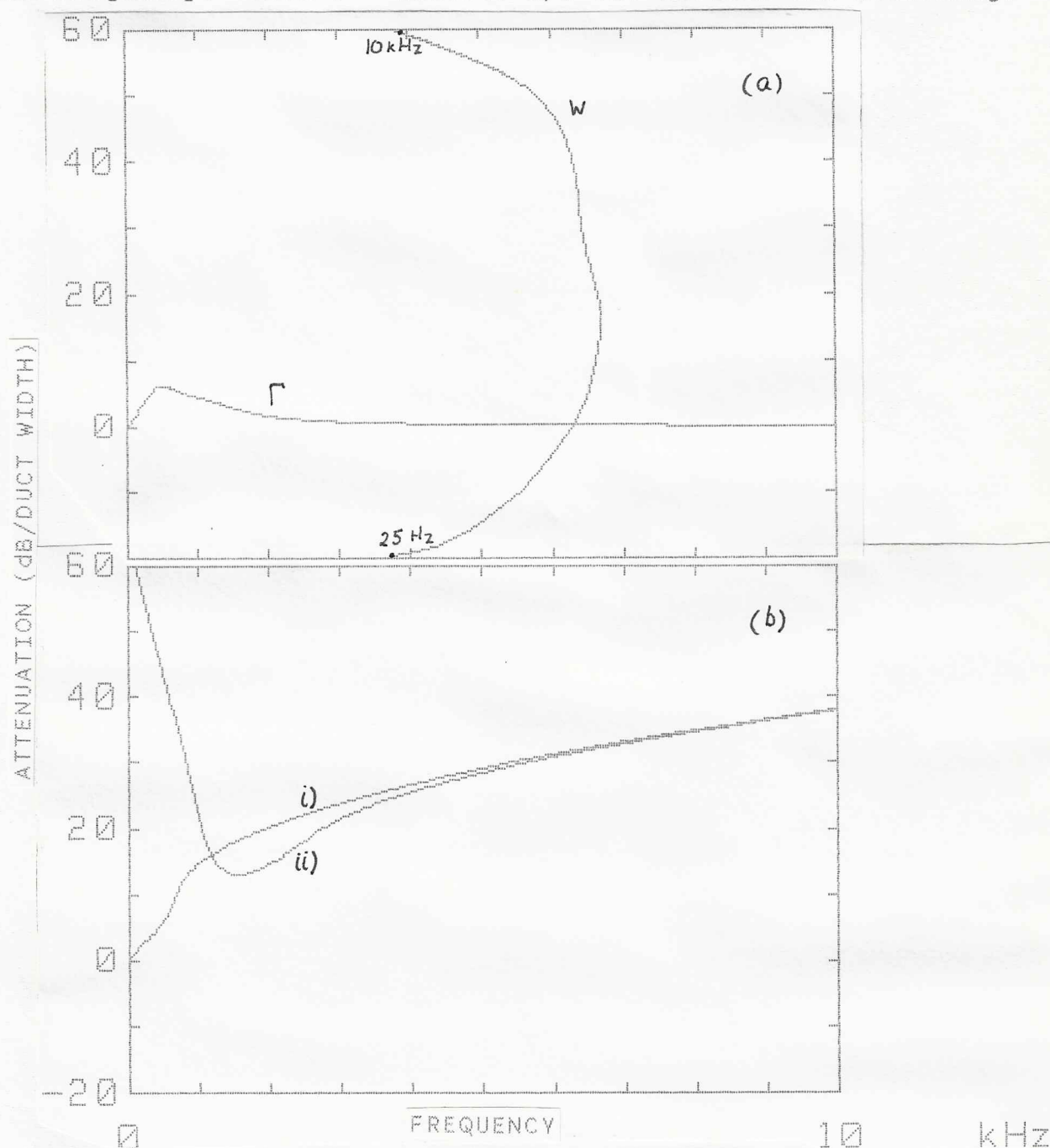


Fig. 2.3. Duct width $l = 0.3 \text{ m}$; lining thickness $d = 0.15 \text{ m}$
 (a) $\sigma = 7000 \text{ N}\cdot\text{s}/\text{m}^4$ and r is the attenuation curve (0,0)
 w is the Nyquist plot.
 (b) $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}^4$; i) is the (0,0) mode and
 ii) is the (0,2) mode 'calculation'.

$$k = 0.365; \quad \Gamma_a = 1.288 + j \cdot 1.362;$$

$$Z_a/\rho c = 2.849 - j \cdot 2.615; \quad M = 0.004 + j \cdot 0.050.$$

As can be seen $M \approx 0$ and Scott's equation can be replaced by Morse's and as with Morse's equation the (0,2) mode begins at $\pi + j \cdot 0$ (as is borne out by the author's own work with 'Extended', see p. 12) and not $\frac{1}{2}\pi + j \cdot 0$ as shown in Fig. 2.2.

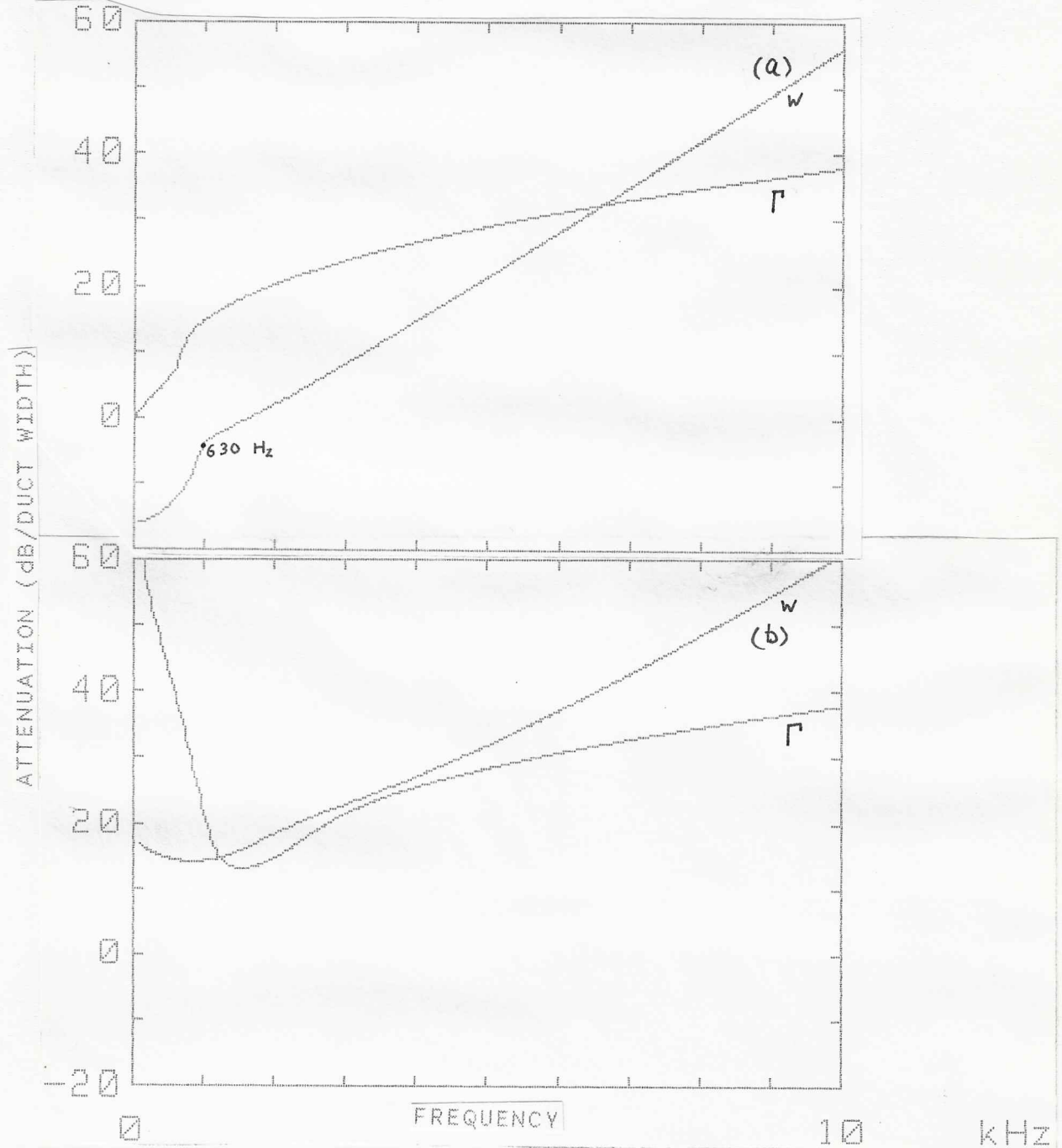


Fig. 2.4. Duct width $l = 0.3$ m; lining thickness $d = 0.15$ m
 (a) $\sigma = 3000$ N·s/m⁴ and Γ is the attenuation curve (0,0)
 w is the Nyquist plot.
 (b) $\sigma = 3000$ N·s/m⁴ and Γ is the attenuation curve (0,2)
 w is the Nyquist plot.

A frequency resolution of 25.19 Hz is employed in Fig. 2.3 (a) and so the starting frequency is also 25.19 Hz and not 20 Hz as in Fig. 2.2. The Nyquist plot in Fig. 2.3 (a) follows that shown in Fig. 2.2 for $\sigma = 7000 \text{ N}\cdot\text{s}/\text{m}^4$. The values of w at low and high frequency were calculated with 'Extended' and found to be:

Mode (0,0); $f = 25.19 \text{ Hz}$; $\text{Re}\{w\} = 0.0052$; $\text{Im}\{w\} = 0.0790$

" $f = 10.00 \text{ kHz}$; $\text{Re}\{w\} = 1.5206$; $\text{Im}\{w\} = 0.0991$.

It can be seen that w starts at $0 + j\cdot 0$ and has a high frequency end point at $\frac{1}{2}\pi + j\cdot 0$ in agreement with the previous discussion. 10 iterations were required for the result at 25.19 Hz whereas only 2 iterations were required at 10.00 kHz, and this with an error bound, $\epsilon = 10$ throughout the frequency interval. Altogether 957 iterations were used to calculate for extended reaction at 397 evenly spaced frequencies from 25.19 Hz to 10.00 kHz, taking about 200 s.

All the calculations in this thesis have been to an accuracy much greater than 4 decimal places. Information is presented with 4 or fewer places of decimals for the sake of convenience.

With the frequency resolution of 25.19 Hz employed in Fig. 2.4 the starting frequency is also 25.19 Hz and not 20 Hz as in Fig. 2.2. In this case 'Extended' with $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}$ gives the following:

Mode (0,0); $f = 25.19 \text{ Hz}$; $\text{Re}\{w\} = 0.0064$; $\text{Im}\{w\} = 0.0787$

Mode (0,2); $f = 25.19 \text{ Hz}$; $\text{Re}\{w\} = 3.1406$; $\text{Im}\{w\} = 0.0001$.

The low frequency starting value for w (0,2) mode is $\pi + j\cdot 0$ and not as shown in Fig. 2.2.

In each example here, 9 iterations were used. For calculations with the (0,0) mode this number rapidly fell to 3 and then to 2 iterations per frequency. On the other hand, for calculations with the (0,2) mode the number of iterations steadily increased until 95 were required to obtain a solution at 1,209.12 Hz. From this frequency up to 1,410.64 Hz, 101 iterations were performed (see the discussion in § 3) at the 8 frequencies in this interval. After this the algorithm continued the calculation very rapidly (2 iterations per frequency) using the alternative iteration (Newton-Raphson). The cut on frequency for a 'hard walled' duct with duct width $l = 0.3 \text{ m}$ as in the example above is 1.15 kHz i.e. just below the frequency where difficulties arise in obtaining solutions for the (0,2) mode. The author has experienced many similar situations during the course of this study.

Examination of Fig. 2.2 and Fig. 2.4 (a) reveals that for $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}$ the Nyquist plot of w for the (0,0) mode does not approach $\frac{1}{2}\pi + j\cdot 0$ but rather increases without limit. This is substantiated by a detailed numerical printout of the array for w .

A similar examination for the (0,2) mode calculations does not reveal any 'jumps' in the root's progress. Difficulties in

converging to a root around cut on for the (0,2) mode lead the author to suspect the results in Fig. 2.4 (b). Comparison however of i) and ii) in Fig. 2.3 reveals two distinct attenuation curves for the two modes that slowly and smoothly approach each other. Using the values for the branch cut in 'Extended' the only two possible solutions are for the (0,0) and the (0,2) modes and one would intuitively expect one or the other mode to show a peak somewhere in the frequency interval chosen. In the example with $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}$, Wassilieff has (0,0) starting as the least attenuated mode (it always is below the first higher order mode cut on) and then switching over at 630 Hz to the (0,2) mode as the least attenuated (see Fig. 2.2). The results with 'Extended' would seem to indicate that the (0,0) mode attenuation increases without limit, however the author is of the opinion that the algorithm requires further testing before making any conclusions regarding the behaviour of the (0,2) mode in this example.

Eq. 2.2.18 can be shown to be equal to eq. (17) given by Mechel in [ref. 12].

With $k = \omega/c$ and $\Gamma_a = j \cdot h$ then eq. 2.2.18 can be written as,

$$\lim_{f \rightarrow 0} Z_a / \rho \cdot c = -j \cdot A \cdot \omega / (2\pi \cdot h \cdot c). \quad 2.2.19$$

From Scott [ref. 4] we recall that the porosity, P , and complex density, ρ' , are given as

$$P = h \cdot K / (\rho' \cdot \omega^2); \quad \rho' = Z_a \cdot h / \omega$$

therefore on eliminating ρ'

$$P = h \cdot K / (Z_a \cdot \omega); \quad \text{giving } \omega/h = K / (Z_a \cdot P).$$

On substitution into eq. 2.2.19 this gives

$$\lim_{f \rightarrow 0} Z_a / \rho \cdot c = -j \cdot A \cdot K \cdot / (2\pi \cdot Z_a \cdot c \cdot P). \quad 2.2.20$$

On multiplying through by $Z / \rho \cdot c$ and taking the square root,

$$\lim_{f \rightarrow 0} Z_a / \rho \cdot c = \left[-j \cdot A \cdot K \cdot / (2\pi \cdot \rho \cdot c^2 \cdot P) \right]^{\frac{1}{2}}. \quad 2.2.20$$

Comparison of eq. 2.2.20 with eq. (17) in Mechel [ref. 12] shows that equivalence is established if $K/\rho \cdot c^2$ equals gamma (γ) the ratio of the the specific heats at constant pressure and volume respectively. This confirms what Wassilieff [ref. 9] has written concerning Kurze and V6r [ref. 18], see the discussion in § 2.2.1 on p. 11.

3. THE ALGORITHM ('Extended')

3.1. Introduction

The purpose of this algorithm is to solve the transcendental equation 2.2.9 in § 2.2 i.e. to find those values of w for which $f(w) = 0$. This achieved it remains to extract the values of Γ both real and imaginary by using the relations in eq. 2.2.8. as is done in programme lines 10750 and 10840

Bearing in mind that w is complex, then $f(w)$, which in turn is also complex, represents a surface in a 4-dimensional space and the expression $f'(w)$ is the equation of another surface which touches $f(w)$ tangentially at the w in question. This happens an infinite number of times for each w and represents the infinite number of modes.

In order to 'picture' this, one of the dimensions has to be 'removed'. Consider for example the modulus $|f(w)|$, if plotted against $\text{Re}\{w\}$ and $\text{Im}\{w\}$, where $w = \text{Re}\{w\} + j \cdot \text{Im}\{w\}$, this would reveal a surface, the contours of which can be written as

$$|f(w)| = C \quad (= \text{a constant}).$$

When $C = 0$ then we have those values in the $(\text{Re}\{w\}, \text{Im}\{w\})$ plane that must necessarily be solutions to $f(w) = 0$. These are just the bottoms of valleys in our 3-dimensional map. Each mode of propagation will have its own valley. How closely these are spaced or whether they cross will influence the the smooth running of the algorithm.

Since Newton-Raphson's method assumes within the intervals considered that we have global convergence [ref. 17; § 2.2] then $f(w)$, $f'(w)$ and $f''(w)$ are continuous. It follows therefore that since $|f(w)| \geq 0$, then $d|f(w)|/dw = 0$ when $C = 0$. This prohibits using the Newton-Raphson method on $|f(w)|$. The above does though let one 'picture' the solutions to $f(w) = 0$ as the locus of all points at the bottom of those valleys which touch the plane $(\text{Re}\{w\}, \text{Im}\{w\})$ see this illustrated in Fig. 2.2 [ref. 9; Fig. 3].

The above is only meant to provide a simple mental picture of what the algorithm is attempting to do geometrically, no consideration has been given here to either possible singularities or branch points.

In Salvadori and Baron [ref. 19; p. 20] we read that finding the complex roots of a transcendental equation leads to the solution of two simultaneous nonlinear equations. In the same reference on p. 46 one can read "...there are no general procedures for the solution of simultaneous nonlinear equations. Initial values of the real unknowns are usually located by trial and error or by graphical means. The solution may then be continued by Newton's method of tangents, provided the initial values be sufficiently close to the roots".

The proviso that the initial values be sufficiently close to the

roots is not always easy to fulfil; below is outlined the techniques used in 'Extended' to achieve this. After first experiencing problems with convergence to a higher root, Christie adopted the following method. Starting with solutions to Morse's equation in the low frequency range, as a good approximation to solutions to Scott's equation, the process is continued using the solution for $f(w) = 0$ from the iteration performed at one frequency as the sufficiently close initial value with which to start the iteration at the next frequency, [ref. 5].

This method works with the examples chosen by Christie and in 'Extended' this same method was at first applied but was latter replaced by a faster and more reliable alternative. Programme lines 10910 and 10920 extrapolate linearly from $\Gamma(J)$ to $\Gamma'(J+1)$, where $\Gamma'(J+1)$ is used to provide the new initial value $w'(J+1)$ with which to start the next iteration to calculate $\Gamma(J+1)$.

Attempts to extrapolate quadratically (2nd order) or with polynomials based on the previously calculated values of Γ (3rd up to $J-1$ th order), see [ref. 17; § 8] and [ref. 19; § 2], either failed or greatly increased programming time. This suggests that these give less accurate initial values than does a linear extrapolation or that the method is uneconomical. Attempts to linearly extrapolate using more than just the present and previous values of Γ have not yet been tried.

'Extended' can perform the iterations using either Muller's or Newton-Raphson's method. Programme lines 10110 to 10170 generate the components of two vectors drawn serially between the last 3 values of w in the complex plane. Line 10190 compares the lengths of these vectors. A 'jump' or discontinuity is defined and so identified if the foregoing vector has a length 3 times less than the 'present' vector, this event is then recorded by adding 1 to the tally K5, see programme line 10210.

The discontinuity, defined above, represents convergence to a root belonging to another mode of propagation. In this event the flag K4 is changed so as to allow another attempt using the alternative iteration, and this event too is recorded, in the array, T(*), see programme lines 10240 to 10260. The choice of using the ratio 1:3 for comparison of vector lengths was arrived at through trial and error. This value is large enough to produce no false alarms and yet still identify real 'jumps' when the frequency increment is chosen to be sufficiently small (see the Figs. 3.3 (g) and (h)).

There is also a 'fail safe'. If, even using the alternative iterative method a 'jump' still occurs then setting the tally K5 (line 10210) equal to 2, programme lines 10670 to 10820 become operative and 'Extended' now linearly extrapolates from $w(J)$ to $w'(J+1)$, from which $\Gamma(J+1)$ is calculated. This was found to give a smoother attenuation curve than if the extrapolation had been directly from $\Gamma(J)$ to $\Gamma(J+1)$. In this case the next initial value for w with which to start the iteration would be a linear extrapolation from $\Gamma(J)$ to $\Gamma'(J+2)$ which could cause another 'jump'. The next initial value $\Gamma'(J+2)$ in this case is determined using Christie's method, i.e. the value $\Gamma(J+1)$, calculated from

the extrapolated $w'(J+1)$, see lines 10780 and 10790.

After having adopted the technique of linearly extrapolating $\Gamma(J)$ to obtain the value with which to start the iteration at the next frequency, there is no record of the fail safe having been used. 'Jumps' have been noticed, however, when calculating for higher order modes i.e. starting with Muller's method and later switching over to the Newton-Raphson method.

In those cases where convergence is slow or not possible (for example, the values of w and w' may oscillate around the solution, as can happen with Muller's method) then the calculation is terminated after 100 iterations, see lines 9990 and 10590, and the present value of w is taken to be the solution.

The array $T(*)$ has room for up to 400 elements. To allow space for the above information, elements $T(\text{Max}+2)$ and $T(\text{Max}+3)$ are reserved respectively for information about the number of 'jumps' (line 10260) and the total number of iterations used (line 10980). This effectively limits the possible maximum number of evenly spaced frequency intervals, Max , to 397. The elements $T(1)$ to $T(\text{Max})$ contain information about the number of iterations used to obtain convergence (as defined by the error bound) at that particular frequency and information about the type of method employed. Positive numbers identify Newton-Raphson's method whereas negative numbers distinguish Muller's method.

This system provides all the necessary information about how the iteration has progressed throughout the frequency range. For example the number of 'jumps' can be read from $T(\text{Max}+2)$ and the number and position of changes between iterative methods can be seen from changes in sign. These two pieces of information, together, can locate where the jumps have occurred and if and where the fail safe has been used.

Further, it should be noticed that the error bound, ϵ , has a different value when using the Newton-Raphson method than when using Muller's method, lines 9080 and 10470. The iterations proceed at very different rates for the two methods and it was found that the following values were the smallest that would permit convergence:

$\epsilon = 10^{-15}$; Newton-Raphson

$\epsilon = 10^{-10}$; Muller

Using the Newton-Raphson method convergence within the above error bound is normally achieved after 2 or 3 iterations and usually for any number from 10 up to 90 iterations using Muller's method.

The speed of convergence also depends on the incremental increase in frequency. With small incremental steps in frequency, convergence within an error bound of 10^{-10} with a Newton-Raphson approach can sometimes be achieved with 1-2 iterations throughout the whole frequency range, beyond the first two or three frequencies.

Other factors, such as the numerical value of the flow resistance, σ , and the ratio of lining thickness to half the width of the air space, b , (see eqs. 2.1.1, 2.1.2 and 2.2.8) also influence the speed of convergence.

Epsilom, ε , is a value which in the course of the iteration is constant, to call this then the error bound is perhaps something of a misnomer since the actual error remains unknown. There are various ways of defining ε , for example using the 2nd derivative $f''(w)$. This would be inappropriate here for two reasons:

- i) there would be a considerable increase in programming time to calculate $f''(w)$ for each iteration performed at each frequency
- ii) the absolute values of ε cannot anyway be improved beyond those mentioned above.

The two different iterative methods are outlined below.

3.2. The Newton-Raphson method

This method is described fully in both [ref. 17; §§ 2.2 and 2.3] and in [ref. 19; §§ 1.6, 1.17].

The solution of eq. 2.2.9 using the Newton-Raphson method proceeds as follows:

$$w - w = -f(w)/f'(w); n = 0, 1, 2, \dots$$

The iteration is stopped when $w - w \leq \varepsilon$, where ε is the error bound. $f(w)$ and its derivative $f'(w)$ are given below:

$$f(w) = w \cdot \tan w + M \left[w^2 + p^2 \right]^{\frac{1}{2}} \cdot \tan \left[b \left[w^2 + p^2 \right]^{\frac{1}{2}} \right] = 0 \quad 3.2.1$$

$$f'(w) = \left[\frac{2w + \sin 2w}{1 + \cos 2w} \right] + M \left[\frac{w}{\sqrt{(w^2 + p^2)}} \left[\frac{2b \sqrt{(w^2 + p^2)} + \sin \{2b \sqrt{(w^2 + p^2)}\}}{1 + \cos \{2b \sqrt{(w^2 + p^2)}\}} \right] \right] \quad 3.2.2$$

This forms the basis of that part of the algorithm between programme lines 9070 to 10190 and lines 10840 to 10950.

Fröberg in [ref. 17; § 2.2] shows that

$$\varepsilon_{n+1} = \varepsilon_n^2 \cdot f''(w) / (2 \cdot f'(w))$$

which means that if $f''(w)/(2 \cdot f'(w))$ is at least not too large then the number of correct decimals is approximately doubled at every iteration.

3.3. Muller's method

This method is described fully in [ref. 17; § 2.5].

By writing $f(w) = f(w) - g(w) = 0$, then the solution of eq. 2.2.9 using Muller's method proceeds as follows:

(N.B. $f(w)$ is not $\text{Re}\{f(w)\}$ and $g(w)$ is not $-j \cdot \text{Im}\{f(w)\}$.)

The two possible iterations are as given below, where $n = 0, 1, 2, \dots$

$$\text{i)} \quad w_{n+1} = f^{-1}(g(w))$$

$$\text{ii)} \quad w_{n+1} = g^{-1}(f(w))$$

$$\text{Letting } f(w) = w \cdot \tan w \text{ and } g(w) = -M \left[w^2 + p^2 \right]^{\frac{1}{2}} \cdot \tan \left[b \left[w^2 + p^2 \right]^{\frac{1}{2}} \right]$$

3.3.1

and using the first of the possible iterative procedures, allows a convergent series of values for w_n to be generated. This forms the basis of that part of the algorithm between programme lines 10290 to 10450. The values so obtained for w_n are tested, as described previously, by programme lines 10470 to 10660.

The expression $f(w)$ can be 'broken up' in other ways but it has been found through trial and error that other combinations lead to very rapid divergence.

The calculation is terminated when $w - w \leq \epsilon$, or when the number of iterations at the particular frequency exceeds 100, see line 10590. The error bound, ϵ , is as mentioned previously a constant given by $\epsilon = \epsilon_n = 10^{-10}$, programme line 10470.

The above procedure can be described using the following:

$$\text{let } f^{-1}(g(w_n)) = z$$

then from 3.3.1. $w_{n+1} = \arctan z$,

$$\text{and } \arctan z = \frac{j}{2} \cdot \ln \left[\frac{j + z}{j - z} \right],$$

3.3.2

where $z = Y1 + j \cdot Y2$, see programme line 10290. Using this notation allows $j + z$ and $j - z$ to be written as,

$$j + z = J1 \cdot e^{J2}; \text{ where } J1 = \left[(Y1)^2 + (Y2 + 1)^2 \right]^{\frac{1}{2}}, J2 = \arctan \left[\frac{Y1}{Y2 + 1} \right]$$

3.3.3

and

$$j - z = J3 e^{J4} ; \text{ where } J3 = \left[(Y1)^2 + (1 - Y2)^2 \right]^{\frac{1}{2}}, J4 = \arctan \left[\frac{-Y1}{1 - Y2} \right]$$

3.3.4

It is important that the + and - signs are kept as above otherwise the arctangent will only give values lying between $+\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. This would be the case if one wrote $Y1/(Y2-1)$ instead of $-Y1/(1-Y2)$.

The arctangent is so defined that to both, principal values, $J2$ and $J4$ can be added integer multiples of 2π . For example one could write $J2 + 2m\pi$ for $J2$ and for $J4$ one could write $J4 + 2n\pi$ such that $J2 - J4 = 2k\pi$, where $k = 0, 1, 2, \dots$. It has not however been possible to obtain convergence using $k = 0$. Solutions to the (0,2) and (0,4) modes have been obtained with $k = 1$ and 2 respectively. Before the refinements of linear extrapolation were introduced some parts of the attenuation curve for (0,0) were found to have been calculated using Muller's method.

This allows eq. 3.3.2 above to be written as,

$$\arctan z = k\pi + (J4 - J2) + j \ln(J1/J3) = Y3 + j \cdot Y4$$

see programme lines 10430 and 10450. Where the value of k (see also line 8980 where $K3$ is used for k) determines which branch one is on. This possibility does not exist with the Newton-Raphson method. It has been found however that Muller's method can be used to start the calculation for a particular mode and that the Newton-Raphson method can continue the process once stability has been established. By stability is meant that the number of iterations per frequency required for solution has fallen to a minimum and that the calculations are progressing smoothly, this is something one can only know from hindsight. These terms have a somewhat qualitative ring about them, the author makes no apologies and asks the reader to bear in mind the quotation from [ref. 19] at the beginning of this chapter.

3.4. Calculations

It was decided to present the results of calculations in this subsection of The ALGORITHM 'Extended' and not separately in another section. The author took this decision since the results presented in the figures below demonstrate the behaviour of some of the properties integral to the algorithm and the difficulties outlined above.

Fig. 3.1 is a repetition of Fig. 1.2 (d) but with a different frequency resolution. A less fine resolution requires less programming time but then the curves produced are less accurate as can be seen.

Fig. 3.2 extends the analysis presented in Fig. 1.2 (d) to both higher and lower values of the flow resistivity. It is noticeable

that for these duct dimensions ($l = 0.5334$ m and $d = 0.1524$ m) the (0,0) mode seems to show 'anomalous behaviour', outside the range $3,000 \text{ N}\cdot\text{s/m} < \sigma < 128,000 \text{ N}\cdot\text{s/m}$, in that the attenuation continues to rise without limit with increasing frequency.

The series of figures in Fig. 3.3 show a parameter study of curve no. 9 in Fig. 3.2. It is demonstrated for $\sigma = 32,000 \text{ N}\cdot\text{s/m}$ (for the duct dimensions above) that the (0,2) mode is the least attenuated mode above frequencies for which this mode is cut on provided the lining thickness is less than 4 cm.

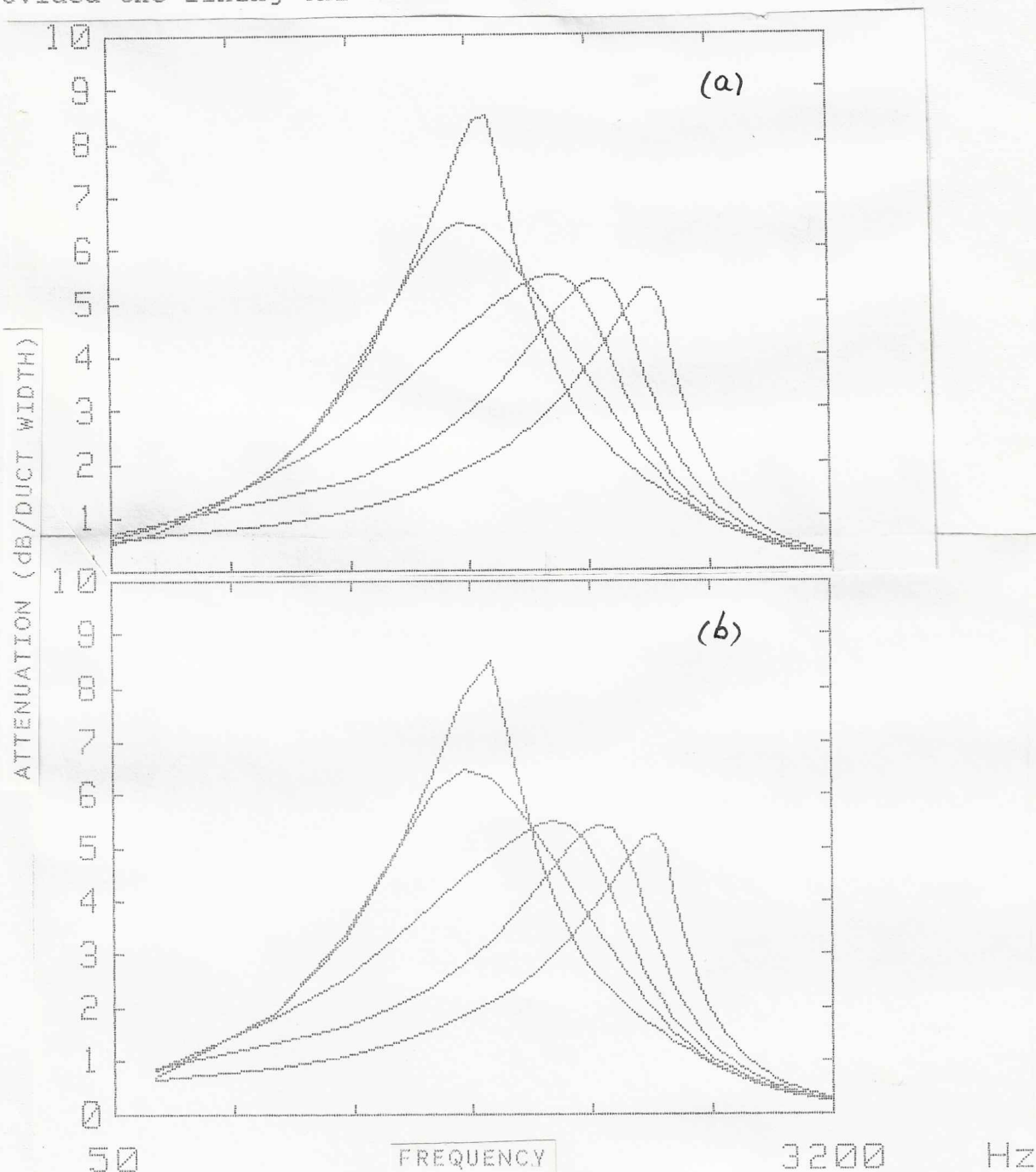


Fig. 3.1. To be compared with Fig. 1.2 (d) where the frequency resolution (F_{res}) is 16 Hz.
(a) $F_{res} = 32$ Hz; (b) $F_{res} = 64$ Hz.

Fig. 3.3 (f) is examined in closer detail in Fig. 3.4. It is shown that the two modes ((0,0) and (0,2)) do cross each other and that for increasing frequency the (0,2) mode is the least attenuated.

The influence of choice of frequency resolution is dramatically demonstrated in Fig. 3.3 (g) and Fig. 3.3 (h). In the latter the proper behaviour of the (0,2) mode is brought out. Fig. 3.5 is a closer inspection of Fig. 3.3 (h). The difficulties encountered around the cut on frequency for (0,2) are clear to see.

The behaviour of modes (0,0) and (0,2) for curve nos. 4, 5 and 11 in Fig. 3.2 is examined in Figs. 3.6 and 3.7 and Fig. 3.8.

This analysis is continued in Fig. 3.9 for curve no. 12 in Fig. 3.2. It is demonstrated that again in this case the (0,2) mode is the least attenuated mode above cut on.

This behaviour was unexpected by the author who at first refused to believe these results assuming that the 'error' lay with the algorithm. This behaviour, where the (0,0) mode is not always the least attenuated and which the author describes as anomalous, should be considered in the light of the remarks on p. 5 of this thesis and Wassilieff's comments in [ref. 9].

The four-pole approach to duct acoustics is based on the assumption of plane wave propagation. The consequences to this approach in view of the above has not yet been fully considered.

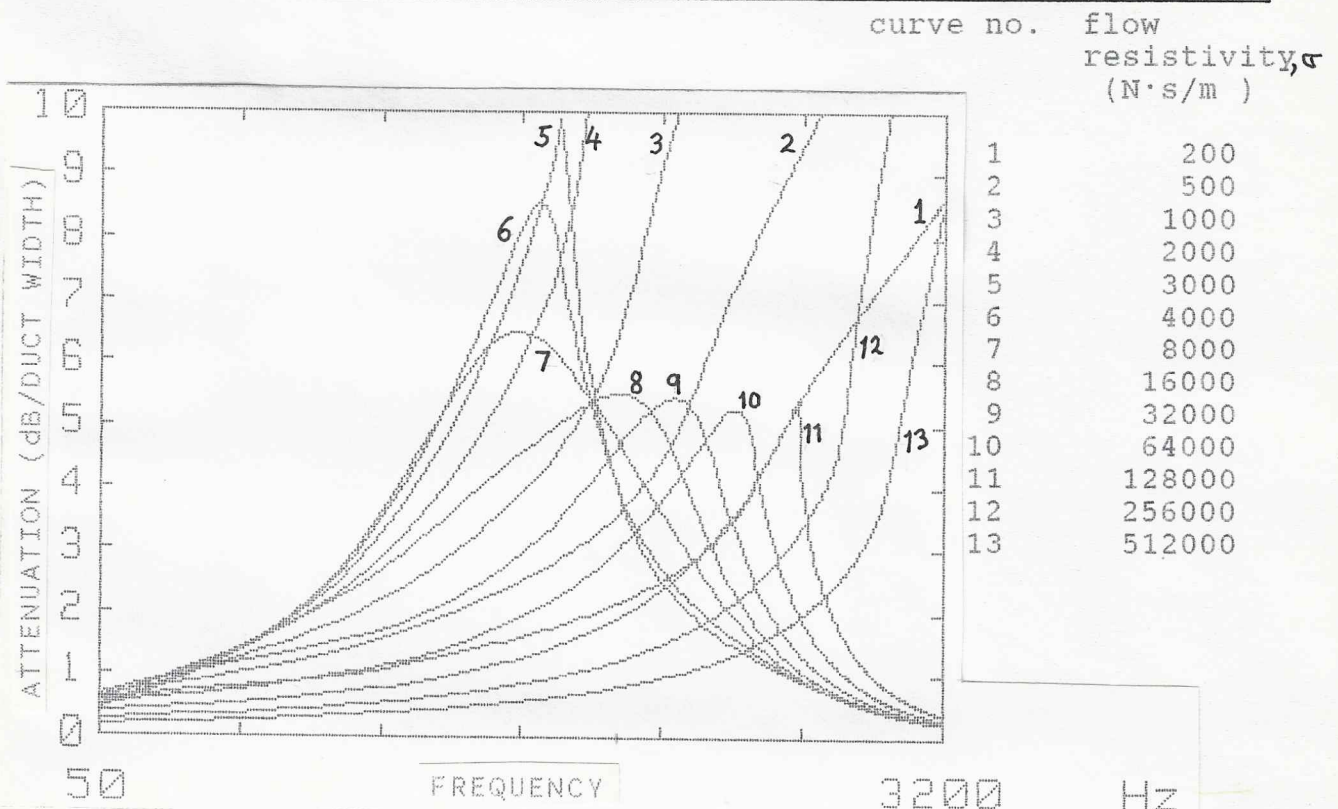


Fig. 3.2. Duct width $l = 0.5334$ m and lining thickness $d = 0.1524$ as in [ref. 5] and Fig. 1.2. The flow resistivity for each curve is as indicated above.

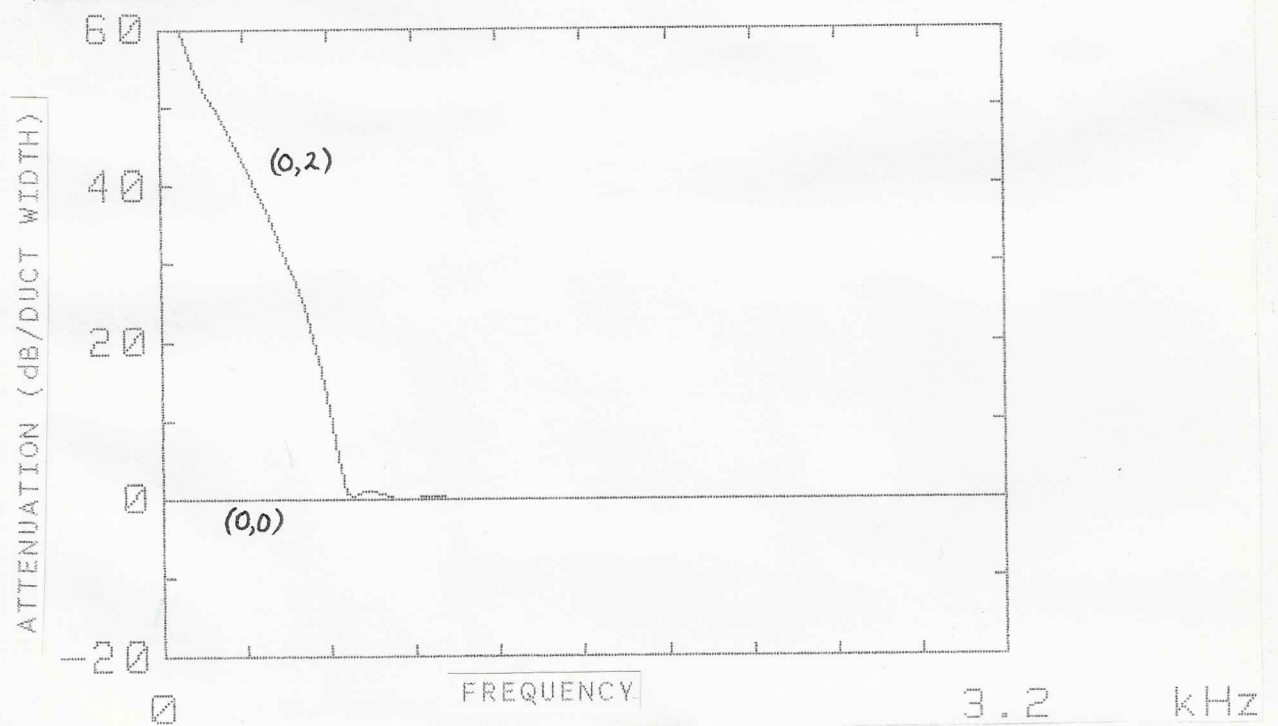


Fig. 3.3. (a) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0001$ m.

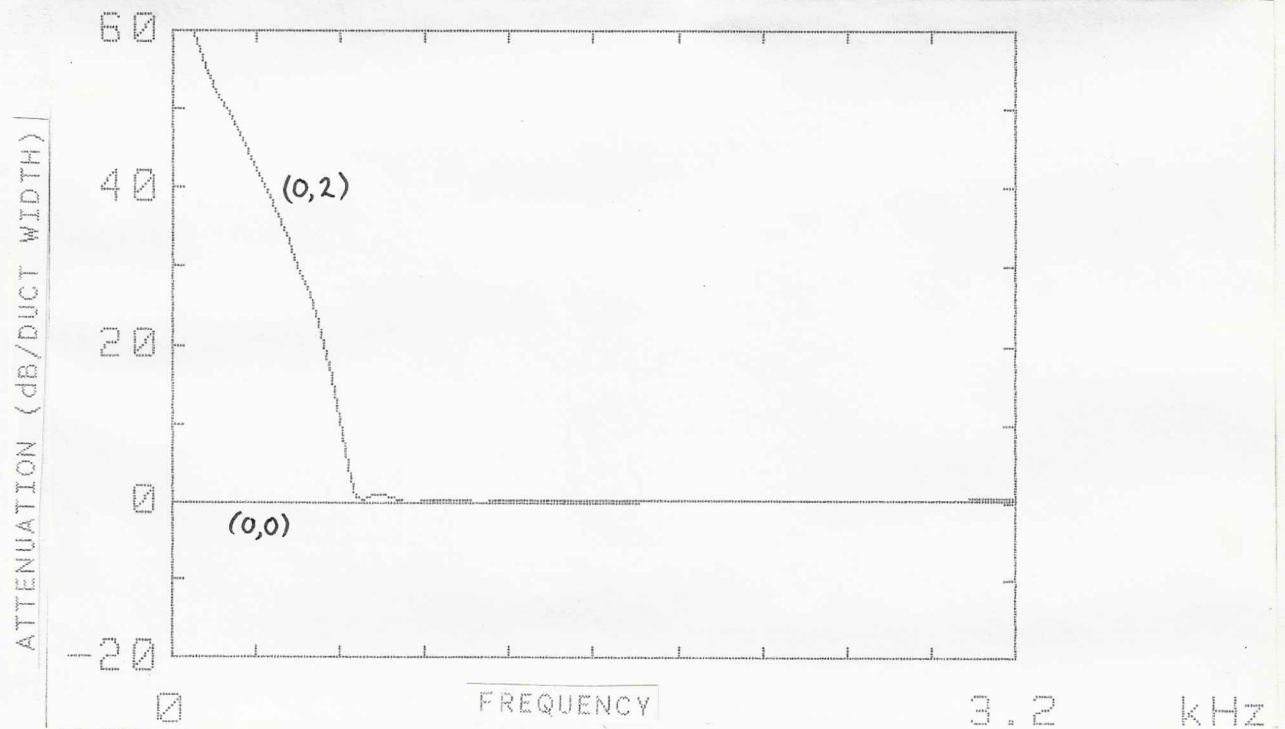


Fig. 3.3. (b) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0010$ m.

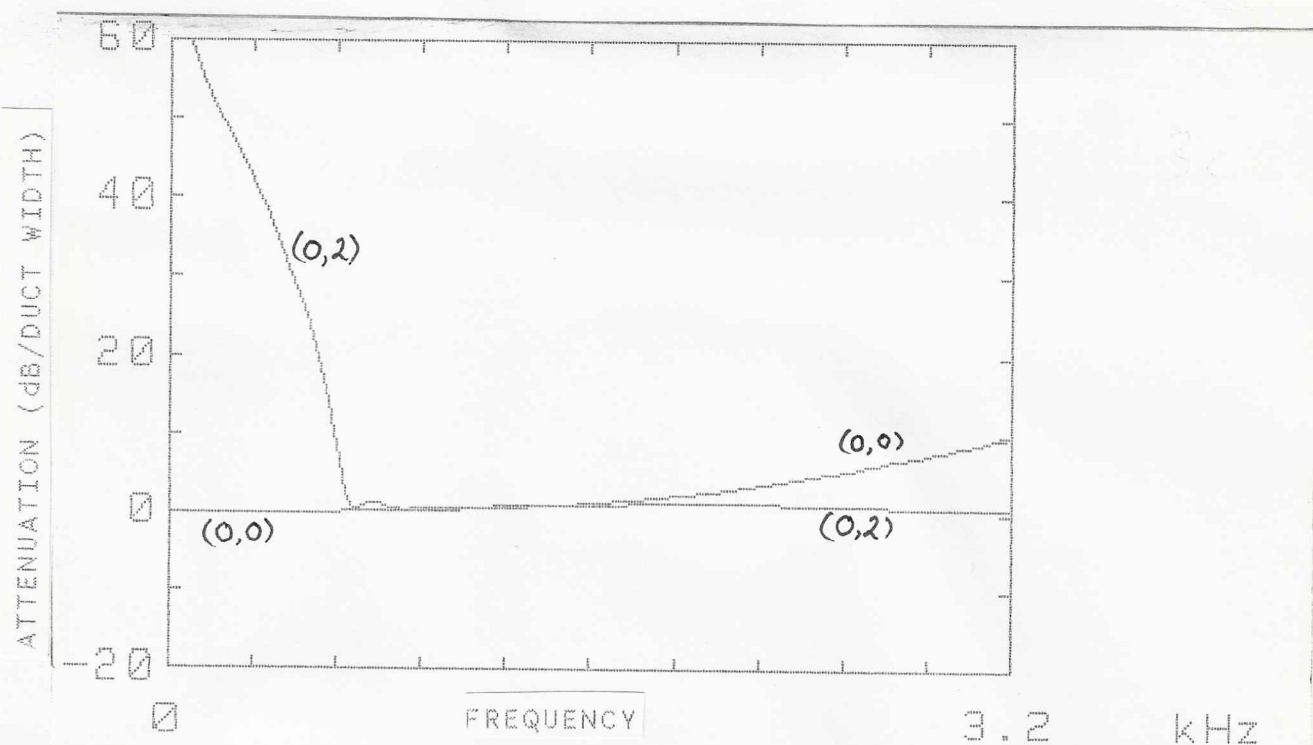


Fig. 3.3. (c) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0050$ m.

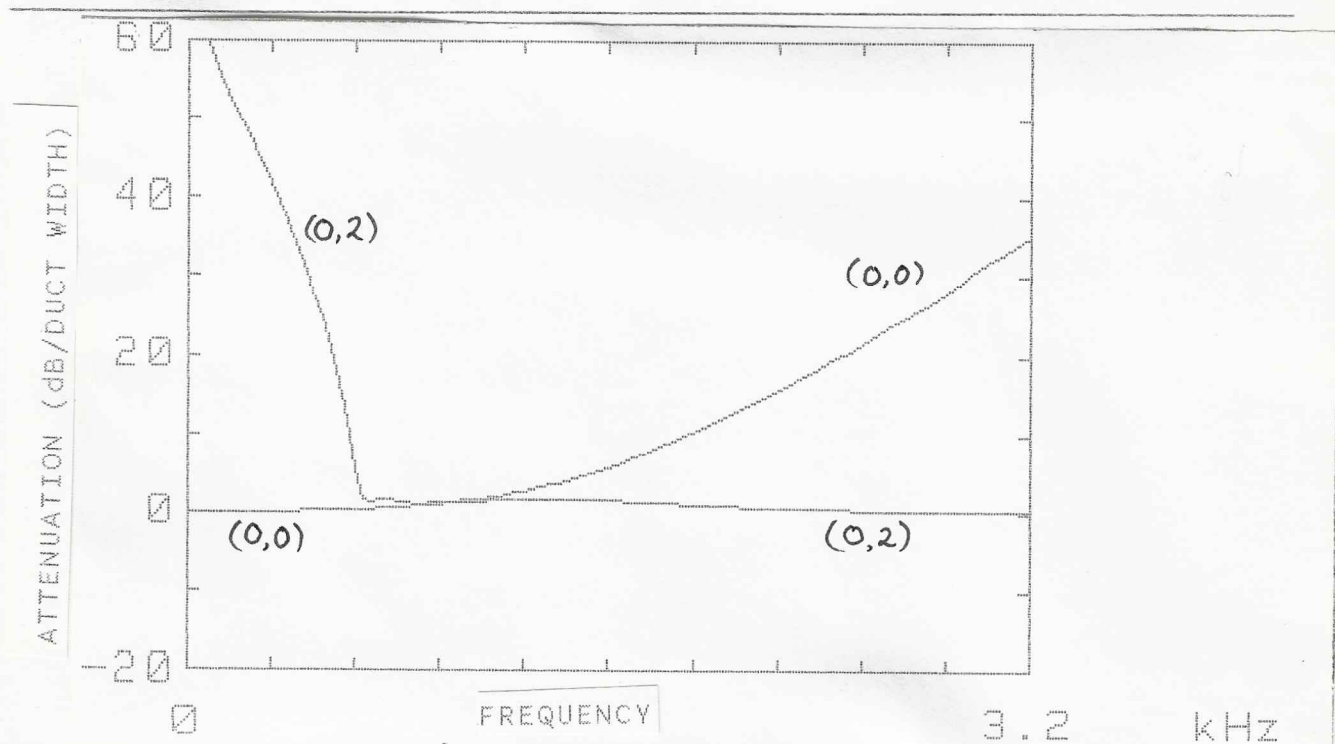


Fig. 3.3. (d) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0100$ m.

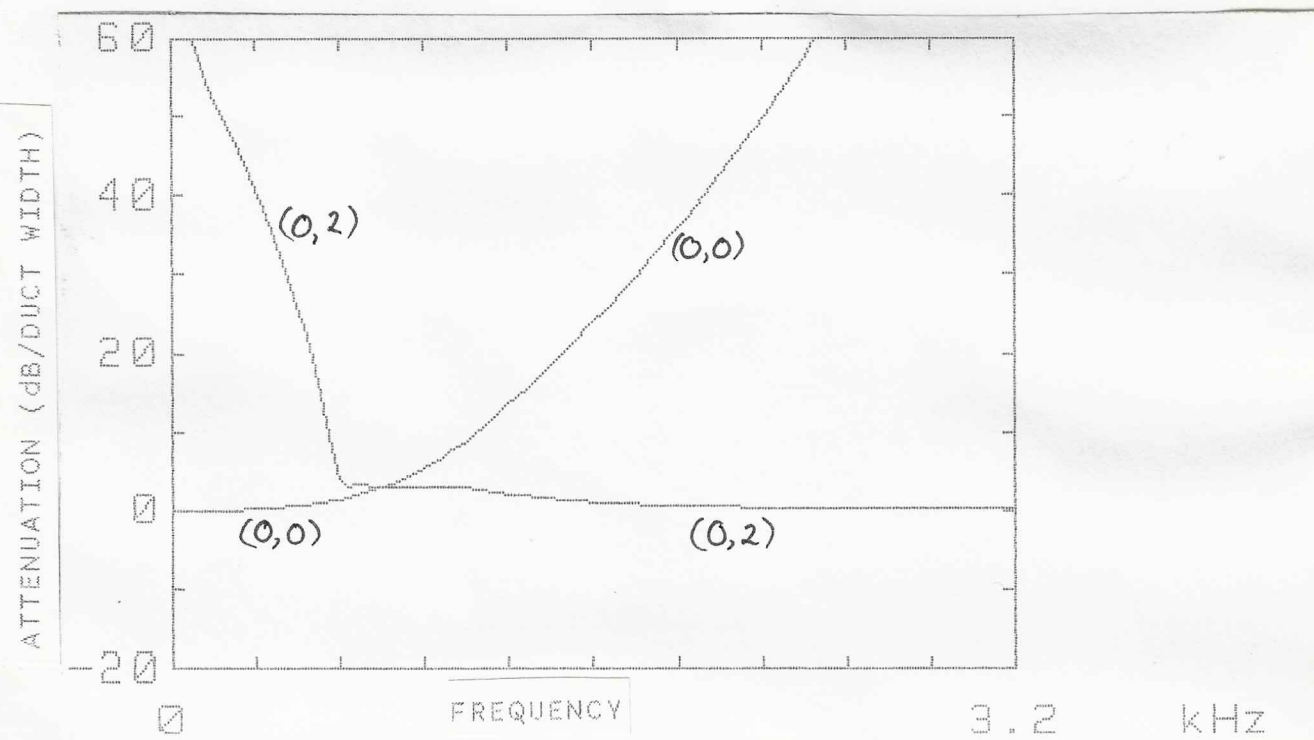


Fig. 3.3. (e) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0200$ m.

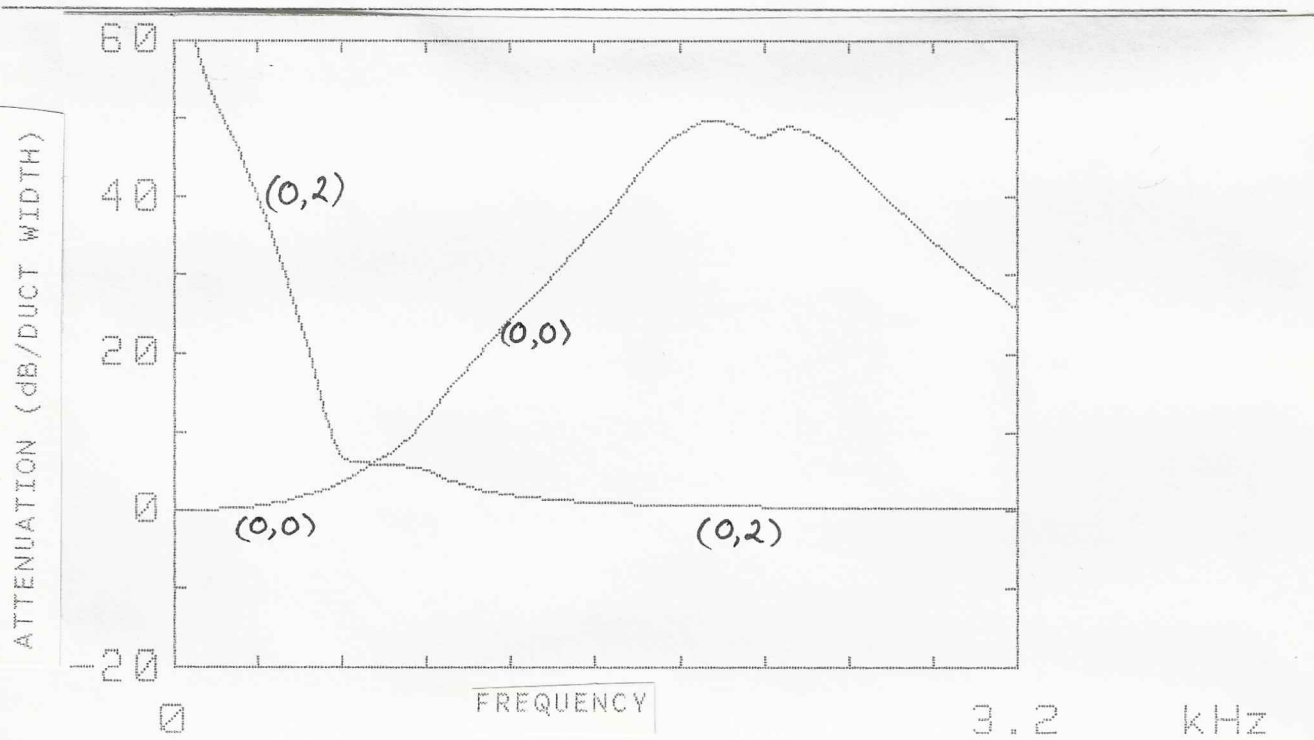


Fig. 3.3. (f) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0300$ m.

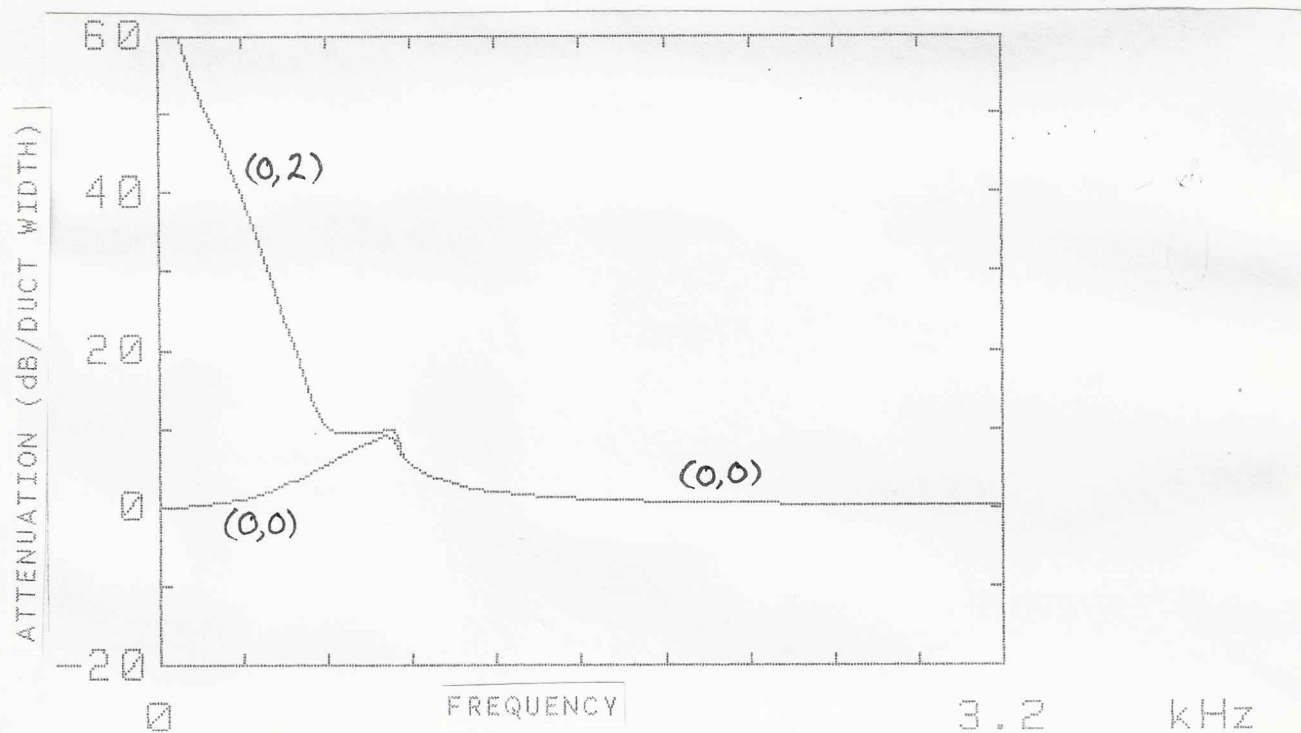


Fig. 3.3. (g) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0400$ m.

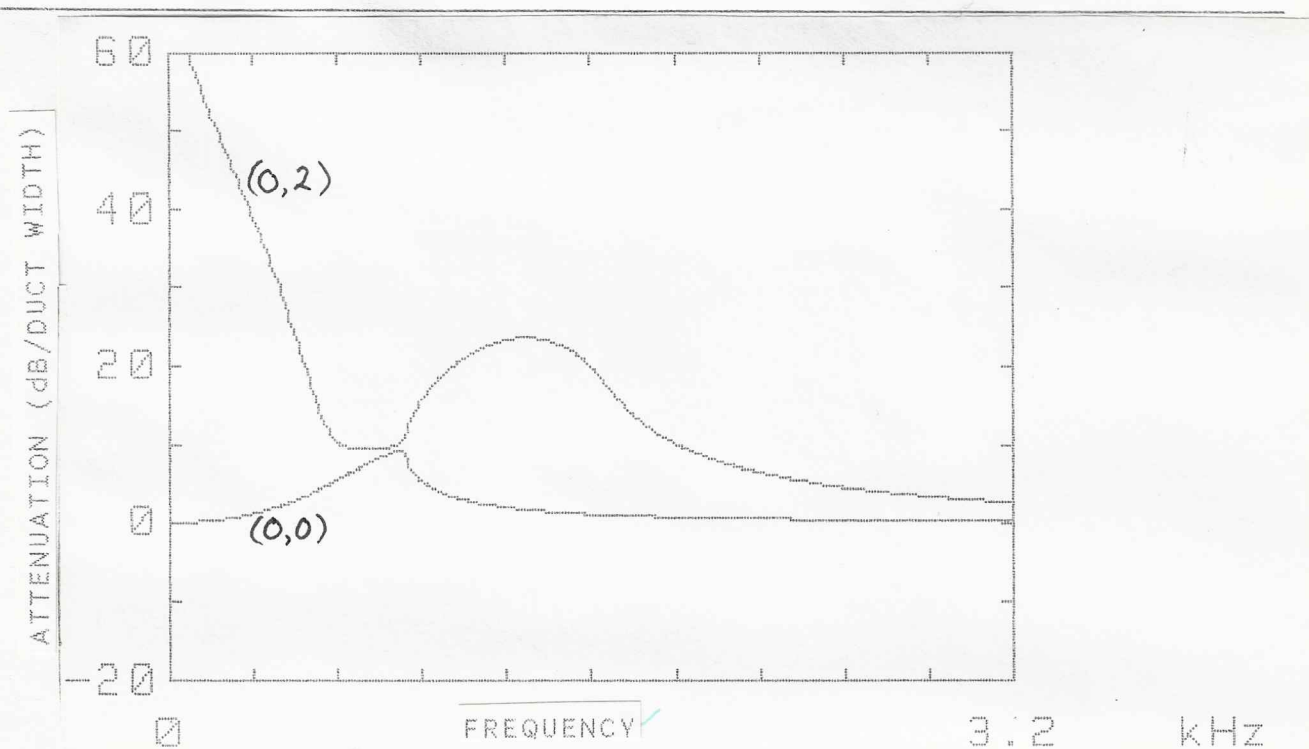


Fig. 3.3. (h) Duct width $l = 0.5334$ m;
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.0400$ m. Freq. res. = 8.06 Hz

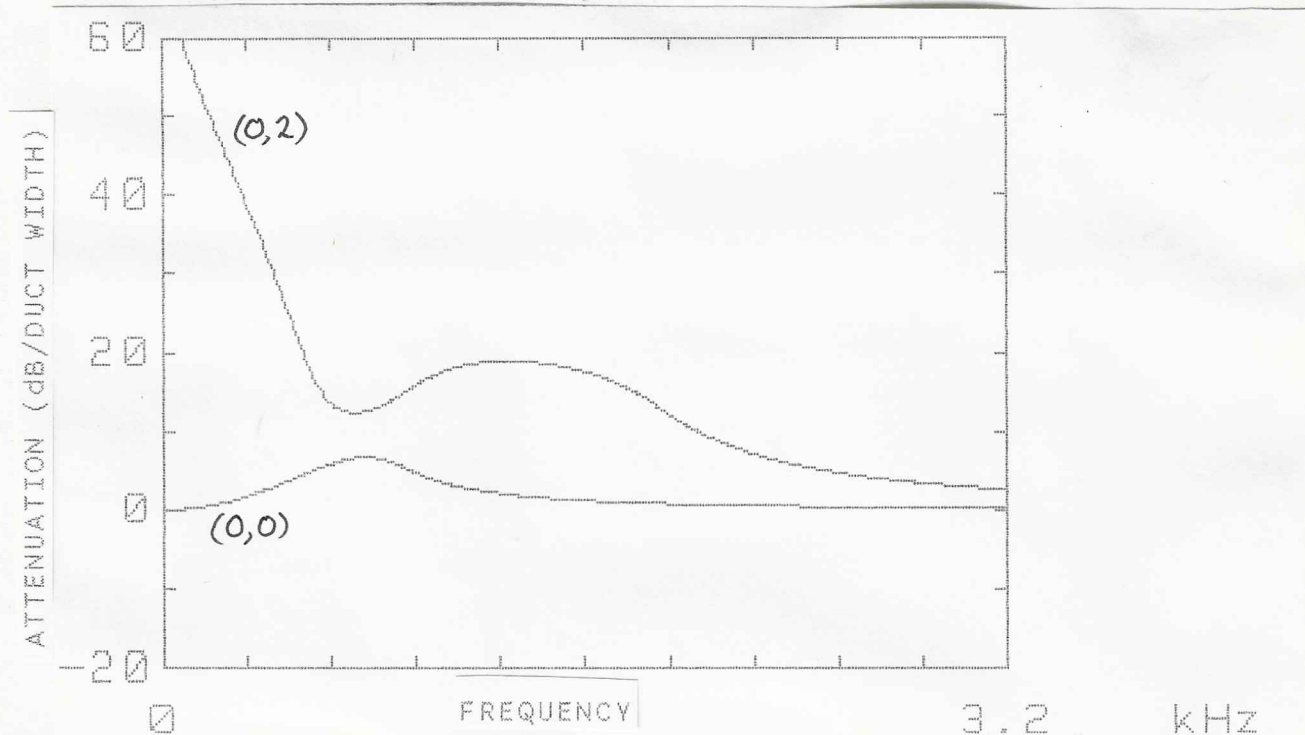


Fig. 3.3. (i) Duct width $l = 0.5334$ m;
Flow resistivity $\sigma = 32,000$ N·s/m .
Lining thickness $d = 0.0500$ m. Freq. res. = 8.06 Hz

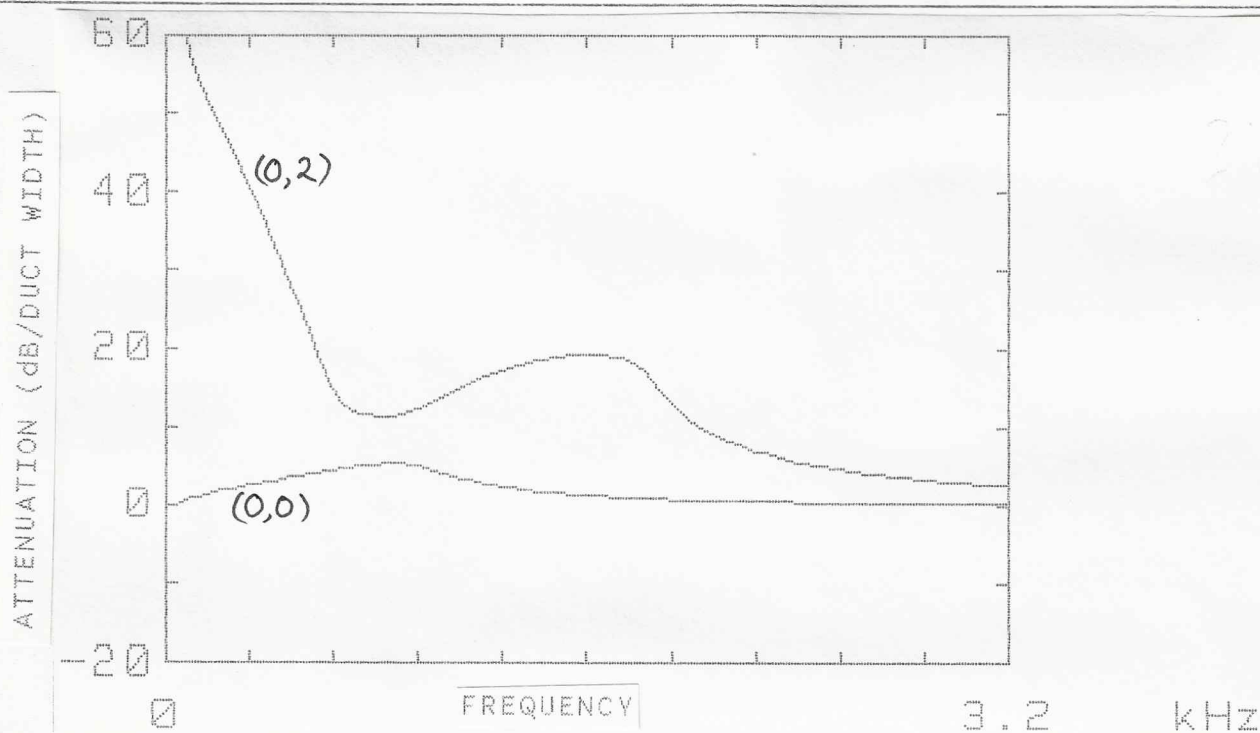


Fig. 3.3. (j) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .
Lining thickness $d = 0.1000$ m.

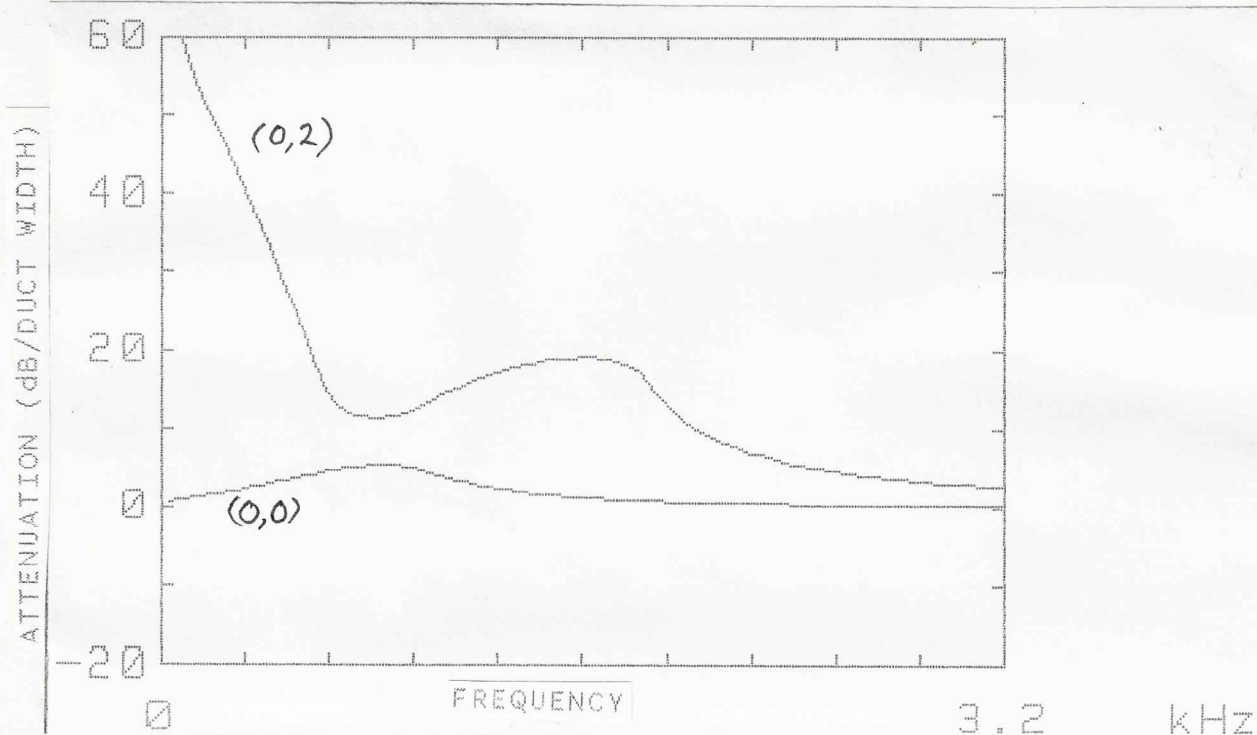


Fig. 3.3. (k) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.2000$ m.

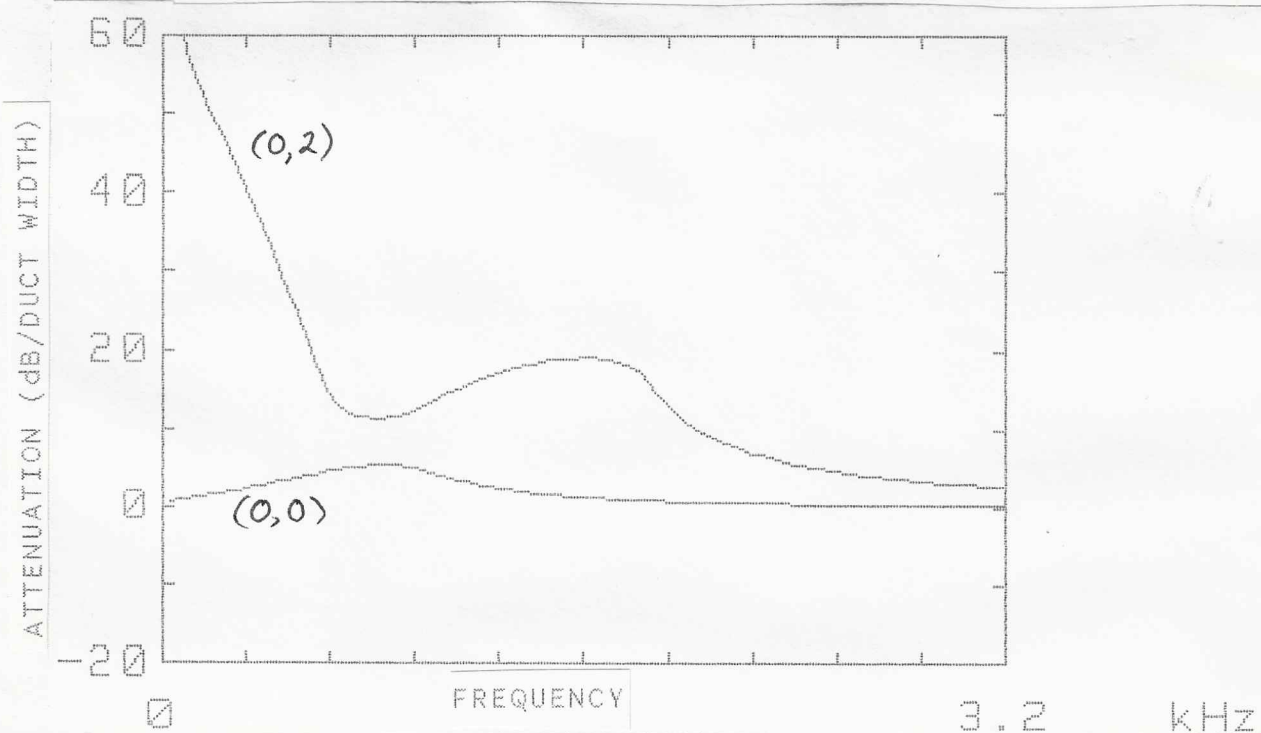


Fig. 3.3. (l) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 0.3000$ m.

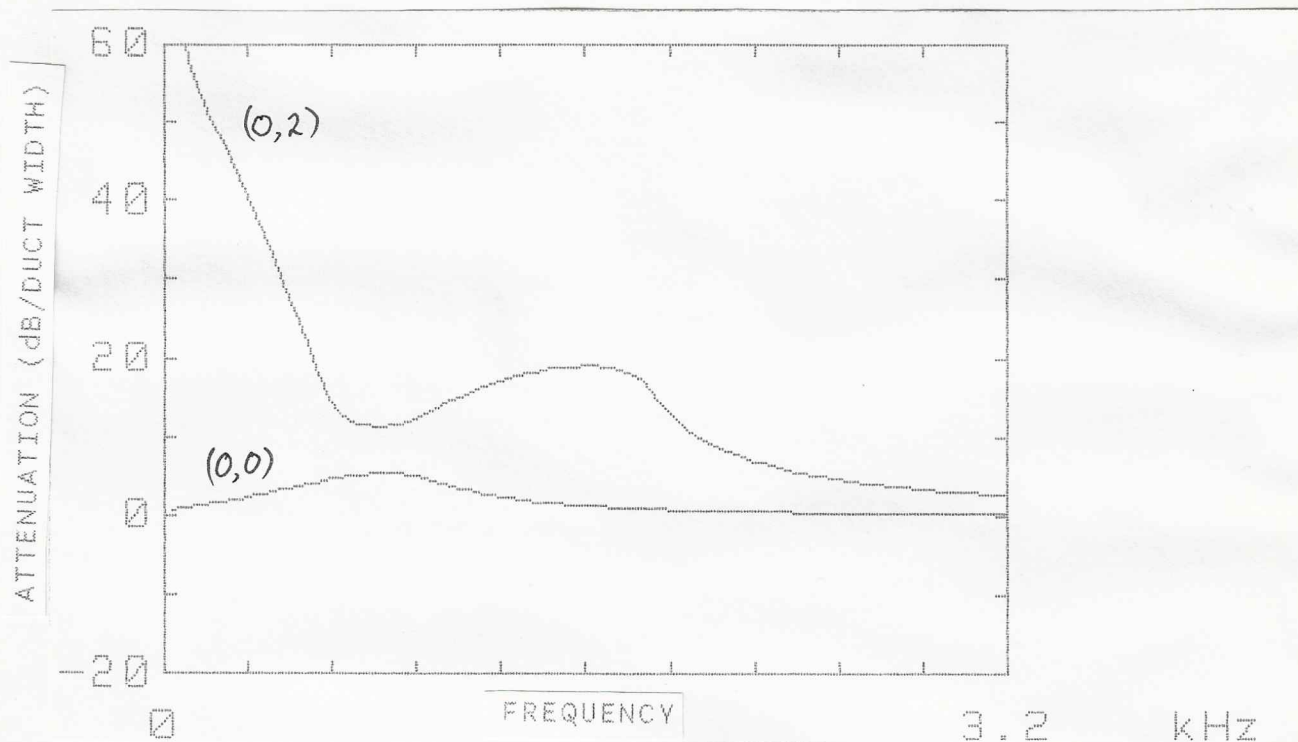


Fig. 3.3. (m) Duct width $l = 0.5334$ m; Freq. resolution = 32 Hz
Flow resistivity $\sigma = 32,000$ N·s/m .

Lining thickness $d = 1.0000$ m.

The attenuation curves in Figs. 3.3 (f), (g) and (h) are now examined in greater detail. Fig. 3.4 shows the curves for (0,0) and for (0,2) with their Nyquist diagrams. A print out of the arrays for w showed the following:

Fig. 3.4 (a) (0,0) mode;

$f = 32$ Hz; $\text{Re}\{w\} = 0.0008$; $\text{Im}\{w\} = 0.0602$

$f = 3.2$ kHz; $\text{Re}\{w\} = 10.6493$; $\text{Im}\{w\} = 1.6016$

Fig. 3.4 (b) (0,2) mode;

$f = 32$ Hz; $\text{Re}\{w\} = 3.1403$; $\text{Im}\{w\} = -0.0027$

$f = 3.2$ kHz; $\text{Re}\{w\} = 1.5658$; $\text{Im}\{w\} = 0.1376$

Negative values for $\text{Im}\{w\}$ as in the example above, occur very seldom and then only very close to zero when using Muller's method. The author believes that this is an example of the 'solution' oscillating about the correct value. In that case the real accuracy (not E_n) can, in this case, be seen to be approximately ± 0.003 . Only when $f > 512$ Hz does $\text{Im}\{w\}$ begin to exceed 0.09 in curve (b).

Further examination of the print out revealed a very rapid decline

in $\text{Im}\{w\}$ (curve (a)) for frequencies just below 3.2 kHz. $\text{Im}\{w\}$ is not approaching $3\frac{1}{2}\pi + j\cdot\frac{1}{2}\pi$ but is in fact approaching $3\frac{1}{2}\pi + j\cdot 0$. According to [ref. 9] $\text{Im}\{w\} \rightarrow 0$ in the limit of increasing frequency and that the high frequency end point for w is always given by [ref. 9]:

$$w = \frac{1}{2} \cdot m\pi + j \cdot 0; m = 0, 1, 3, 5 \dots$$

3.3.5

The author has never found a high frequency end point with $m = 0$, and further the high frequency end point for w is not always given as shown above, see for example curve w in Fig. 2.4 (a) and compare with Fig. 2.2 from [ref. 9].

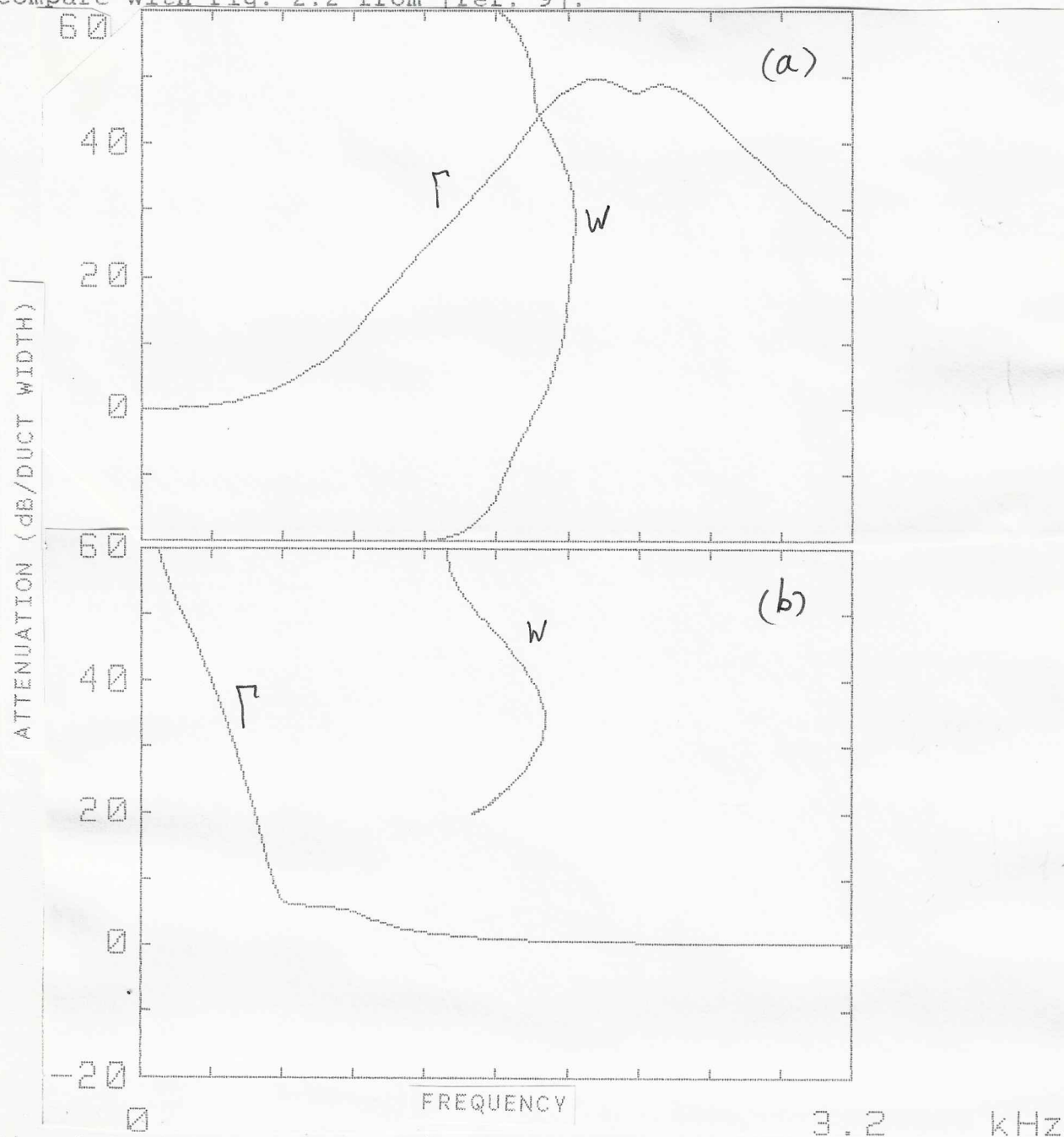


Fig. 3.4. (a) (0,0) mode; (b) (0,2) mode
 Γ attenuation; w Nyquist plot; Freq. res. 32 Hz
 Duct width $l = 0.5334$ m; lining thickness $d = 0.0300$ m
 $\sigma = 32,000$ N·s/m

Fig. 3.4 (a) was produced entirely with the Newton-Raphson method using altogether 235 iterations for 100 frequencies. Likewise Fig. 3.4 (b) was produced entirely with Muller's method using a total of 952 iterations for the 100 frequencies. No problems with either iteration were recorded. The Nyquist plots are smooth, without any sudden deviations. However the double peak in attenuation in Fig. 3.4 (a) is probably due to the presence of yet another higher order mode as is indicated by the high frequency end point value of w . The above also shows that it is the (0,2) mode that approaches $\frac{1}{2}\pi + j \cdot 0$ in the limit. One can only conclude that in this case it is the (0,2) mode that is the least attenuated mode above cut on.

Fig. 3.5 shows the two modes (0,0) and (0,2) in Fig. 3.3 (h) and (g) now separated. A print out of the arrays for w has unfortunately been mislaid but the values were seen by the author to be approaching those below:

Fig. 3.5 (a) (0,0) mode;

$$f = 32 \text{ Hz}; \quad \text{Re}\{w\} = 0; \quad \text{Im}\{w\} = 0$$

$$f = 3.2 \text{ kHz}; \quad \text{Re}\{w\} = \frac{1}{2}\pi; \quad \text{Im}\{w\} = 0$$

Fig. 3.5 (b) (0,2) mode;

$$f = 32 \text{ Hz}; \quad \text{Re}\{w\} = \pi; \quad \text{Im}\{w\} = 0$$

$$f = 3.2 \text{ kHz}; \quad \text{Re}\{w\} = 1\frac{1}{2}\pi; \quad \text{Im}\{w\} = 0$$

No difficulty was encountered with the Newton-Raphson method employed to generate curve (a). The iteration in curve (b) encountered difficulties at 886.60 Hz and from 894.66 Hz the iteration progressed smoothly after switching over to using the Newton-Raphson method. These frequencies correspond to the position in Fig. 3.3 (h) where with the lower frequency resolution of 32 Hz the (0,2) mode attenuation curve followed that for the (0,0) mode. Again it should be noticed that these frequencies are around cut on for the (0,2) mode. The other figures in Fig. 3.3 continue to show the two modes (0,0) and (0,2) as quite separate. No further problems were encountered with the iterations and the low and high frequency values for w remained throughout as those shown above for Fig. 3.5.

It is interesting to note that in this example, increasing the lining thickness beyond 0.1 m has no noticeable effect on the attenuation curves for either the (0,0) or the (0,2) mode.

Neither of the modes shown in Fig. 3.6 is the least attenuated. The Nyquist plot in Fig. 3.6 (b) shows convergence to a higher root, probably (0,4). There is no doubt that again in this example that the (0,2) mode is the least attenuated above cut on. As explained in [ref. 9] and mentioned on p. 14, the least attenuated mode has a high frequency end point for w at $\frac{1}{2}\pi + j \cdot 0$. Only the (0,0) and (0,2) modes can have this value for w as the

high frequency end point. An investigation of the behaviour of the (0,0) mode in Fig. 3.6 (a) does reveal that w appears to increase without limit. Compare the Nyquist plot in Fig. 3.6 (a) with the Nyquist plot for the (0,0) mode shown in Fig. 2.2.

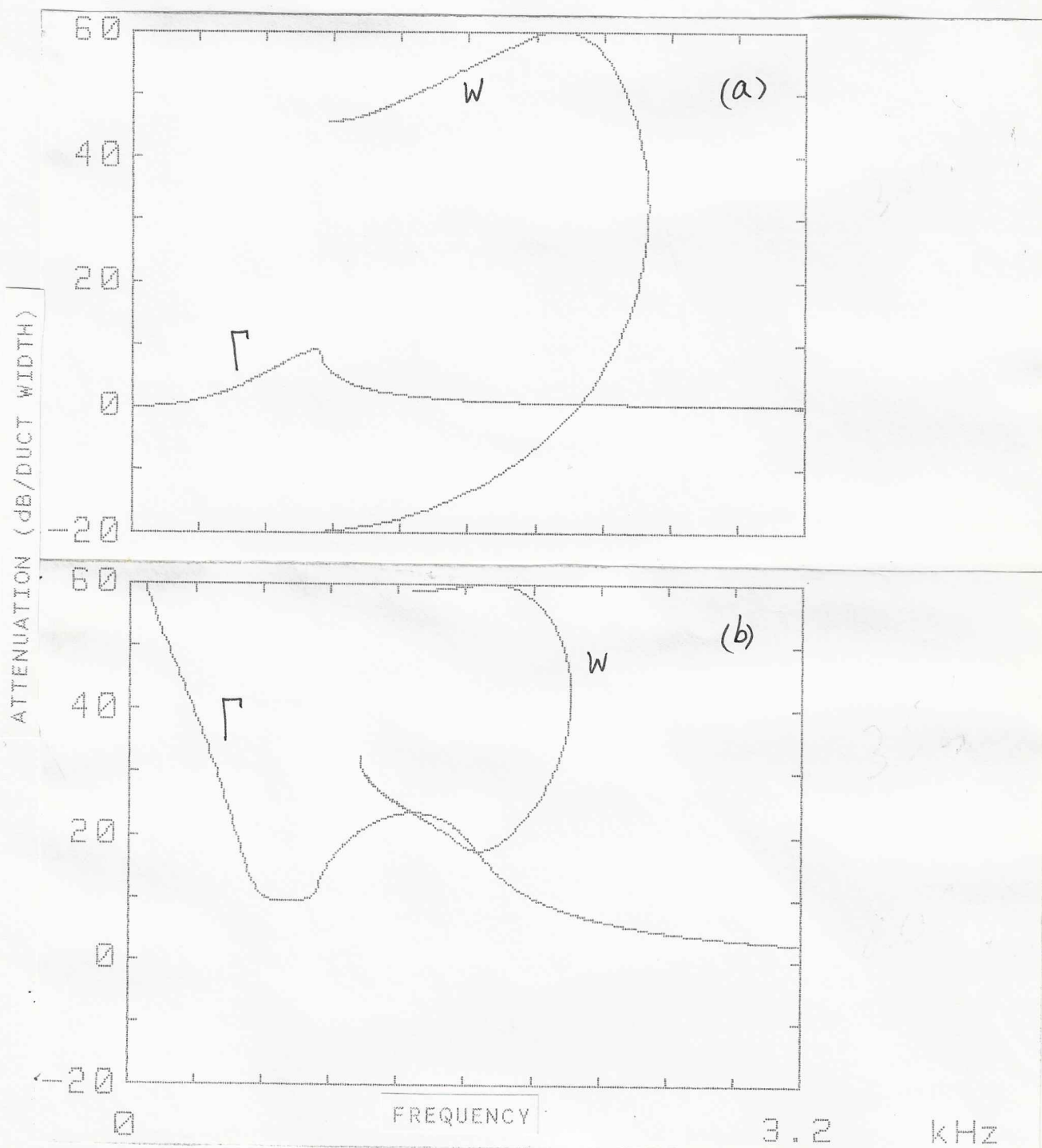


Fig. 3.5. (a) (0,0) mode; (b) (0,2) mode
 Γ attenuation; w Nyquist plot; Freq. res. 8.06 Hz
 Duct width $l = 0.5334$ m; lining thickness $d = 0.0400$ m
 $\sigma = 32,000$ N·s/m

Fig. 3.7 shows that with an increase in σ from 2000 N·s/m to 3000 N·s/m then the (0,0) mode is once again the least attenuated mode. It starts with w at $0 + j \cdot 0$ and possess a high frequency end point of $\frac{1}{2}\pi + j \cdot 0$. The 'peculiar' spiked peak (see

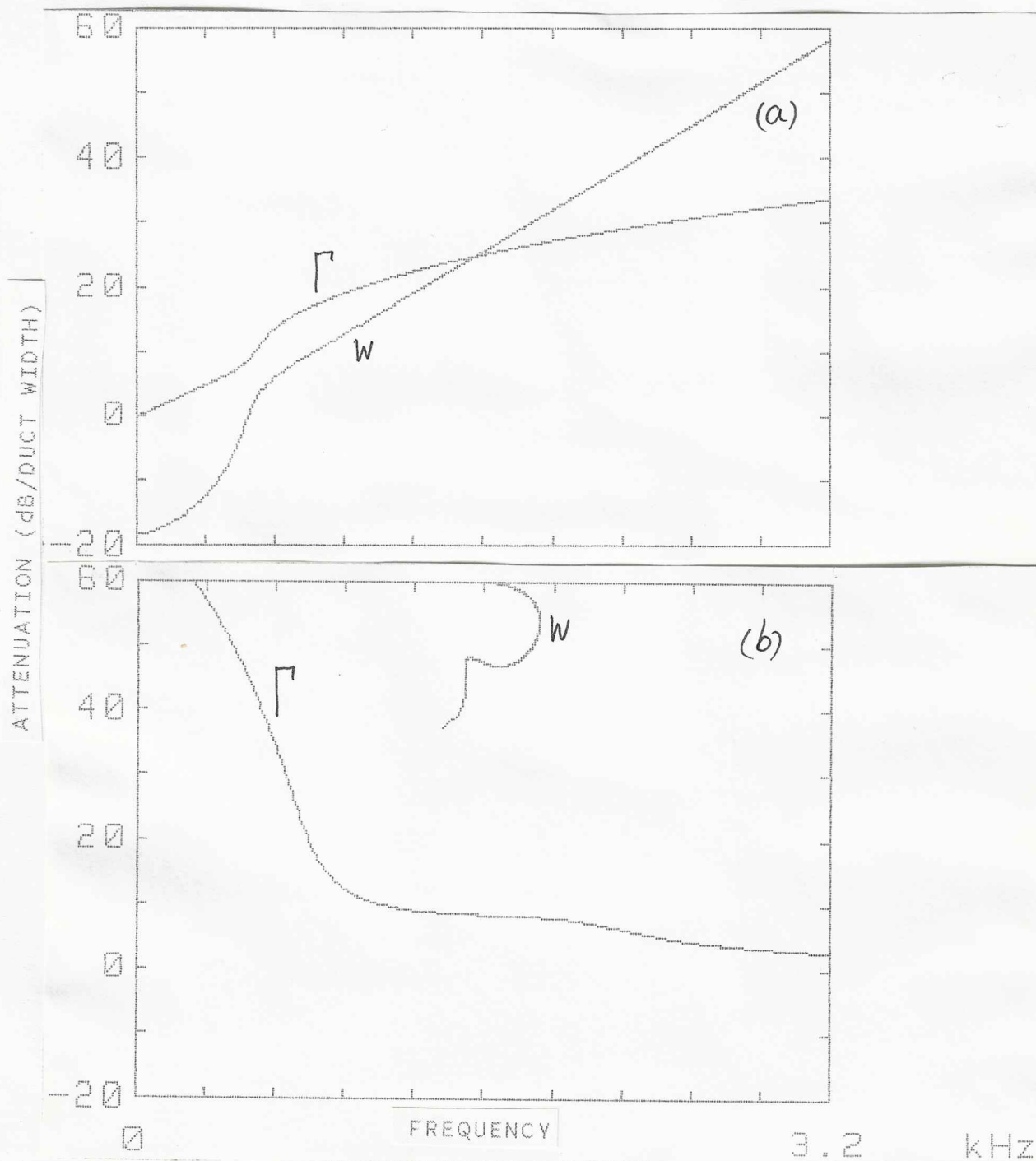


Fig. 3.6. (a) (0,0) mode; (b) (0,2) mode
 Γ attenuation; w Nyquist plot; Freq. res. 8.06 Hz
 Duct width $l = 0.5334$ m; lining thickness $d = 0.1524$ m
 $\sigma = 2,000$ N·s/m . Compare with Fig. 3.2 curve no. 4.

again curve 5 in Fig. 3.2) and the wobble in the Nyquist plot does lead the author to question the validity of this result. It is still possible that the (0,0) calculation has got mixed up with the 'true' roots for the (0,2) mode.

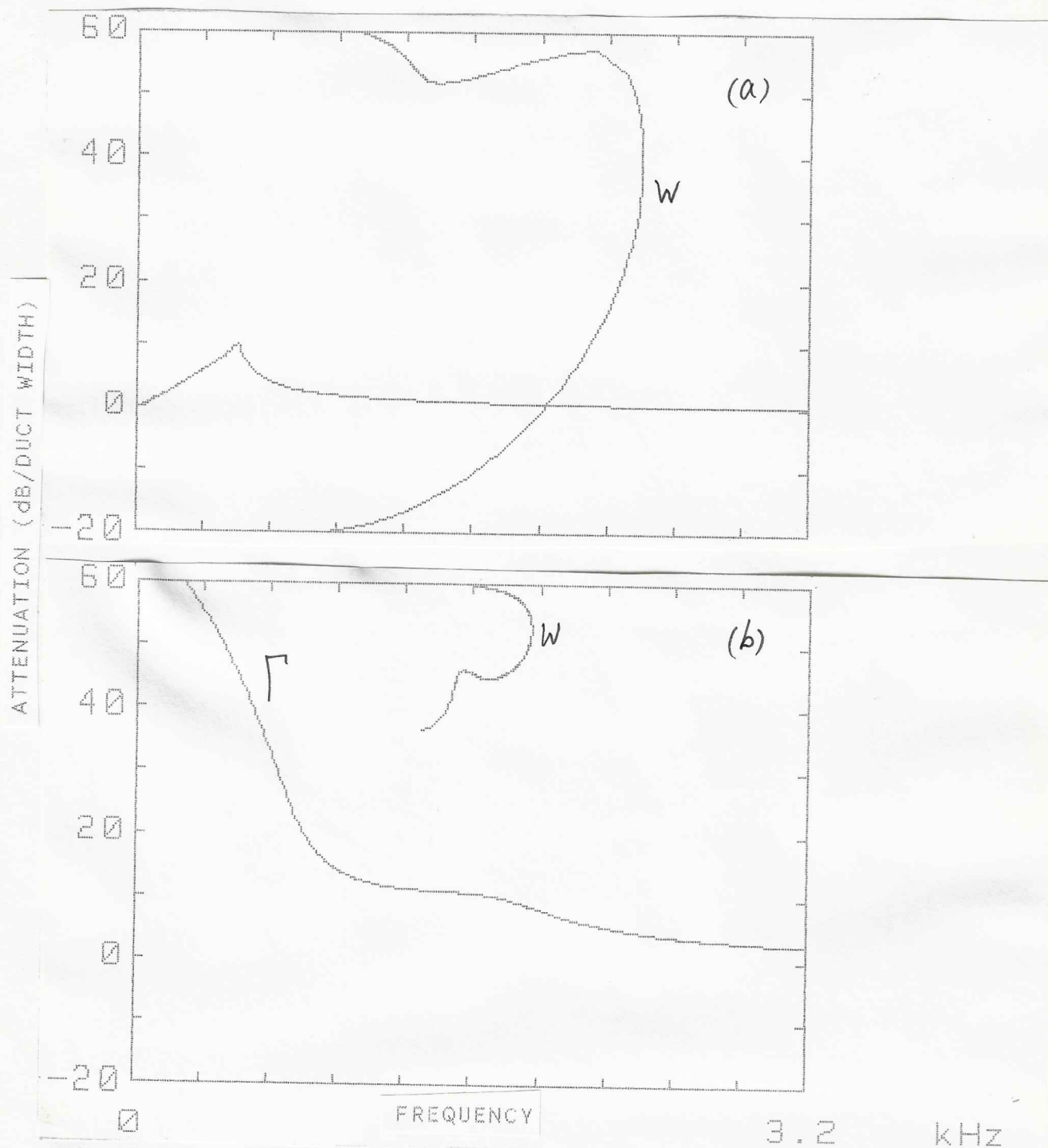


Fig. 3.7. (a) (0,0) mode; (b) (0,2) mode
 Γ attenuation; w Nyquist plot; Freq. res. 8.06 Hz
 Duct width $l = 0.5334$ m; lining thickness $d = 0.1524$ m
 $\sigma = 3,000$ N·s/m . Compare with Fig. 3.2 curve no. 5.

Please note that Fig. 3.8 has a logarithmic frequency scale. In this figure it can be seen that neither the (0,0) nor the (0,2) mode have smooth Nyquist plots. Difficulty in converging was experienced in both plots at 1552 Hz and after this frequency the two run together. At this frequency Newton-Raphson's method for the (0,0) mode required 10 iterations and Muller's method for the (0,2) mode required 101 iterations. Apart from experiencing

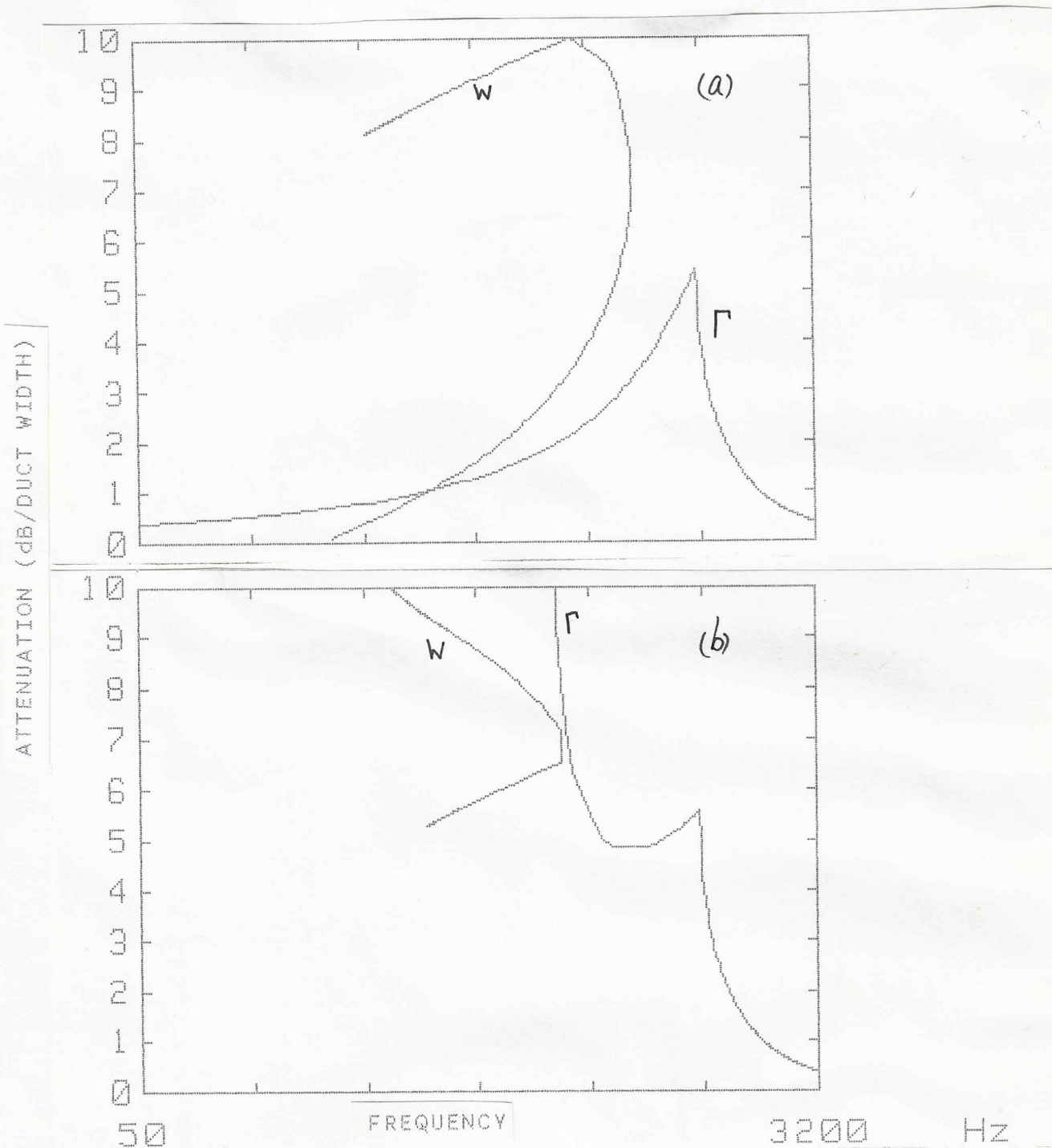


Fig. 3.8. (a) (0,0) mode; (b) (0,2) mode
 Γ attenuation; w Nyquist plot; Freq. res. 16 Hz
 Duct width $l = 0.5334$ m; lining thickness $d = 0.1524$ m
 $\sigma = 128,000$ N's/m . Compare with Fig. 3.2 curve 11.

problems with the first iteration where 101 were required for the (0,0) mode using the Newton-Raphson method the 10 iterations at 1552 Hz were a maximum. Likewise for the (0,2) mode calculations the 101 required at 1552 Hz was also a maximum. The reader may remember from § 3.1 that this number is usually taken as a sign of failure to converge. In each case above the iterations continued with the same method with which the calculations began, i.e. no 'jumps' were registered.

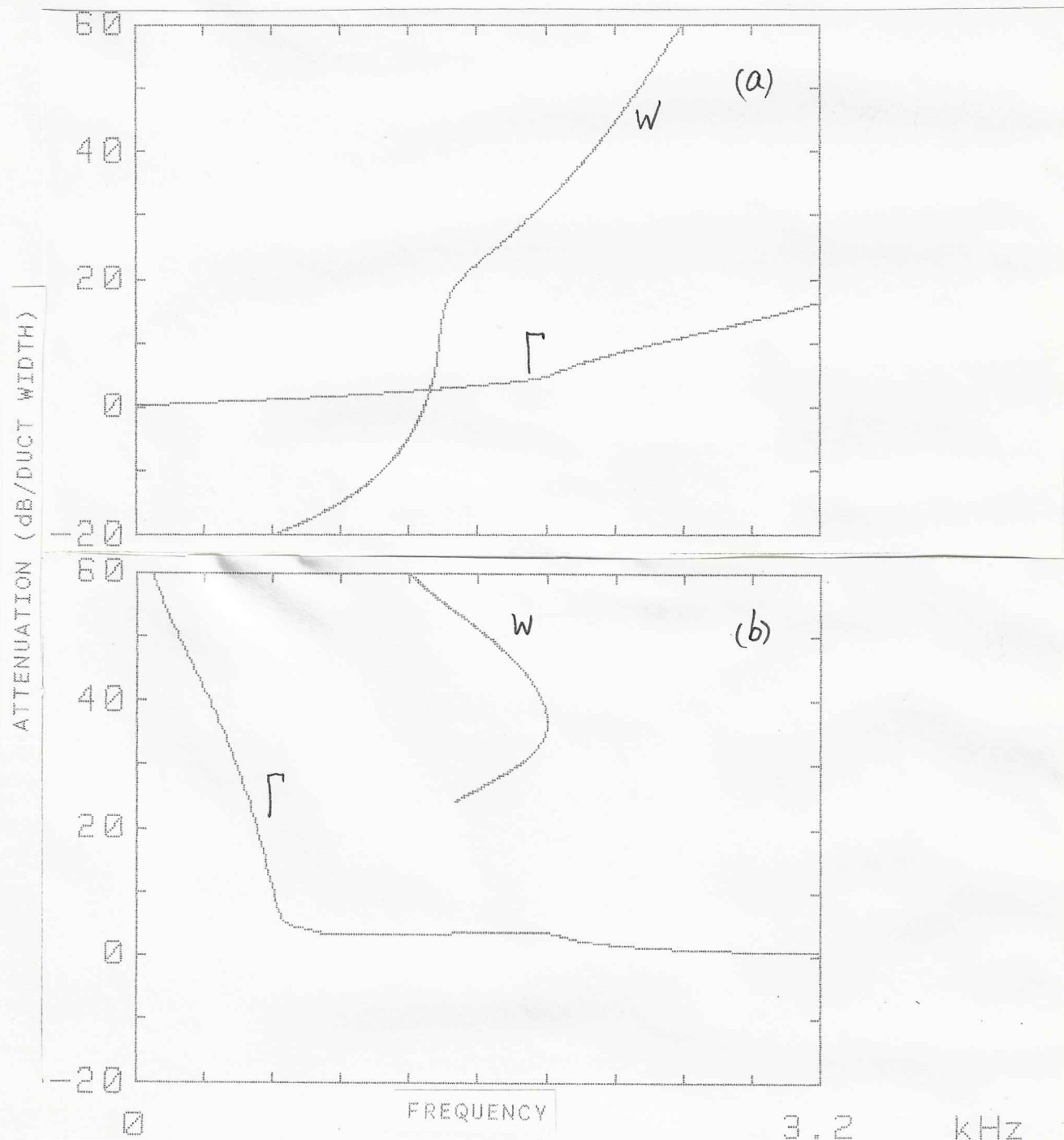


Fig. 3.9. (a) (0,0) mode; (b) (0,2) mode
 Γ attenuation; w Nyquist plot; Freq. res. 8.06 Hz
 Duct width $l = 0.5334$ m; lining thickness $d = 0.1524$ m
 $\sigma = 256,000$ N·s/m . Compare with Fig. 3.2 curve 12.

The plot in Fig. 3.8 (a) is undoubtedly the 'curve' of least attenuation, it starts at $0 + j \cdot 0$ and is approaching $\frac{1}{2}\pi + j \cdot 0$ at 3.2 kHz. Whether this curve consists purely of the (0,0) mode or it continues along the (0,2) mode after 1552 Hz is not possible yet for the author to comment on. The same caution should be shown here as is expressed on p. 19 concerning Fig. 2.4 (b).

Fig. 3.9 displays Nyquist diagrams that are smooth and a closer investigation of their behaviour reveals that w approaches the values shown below:

Fig. 3.9 (a) (0,0) mode;

$$\begin{aligned} f &= 8.06 \text{ Hz}; & \text{Re}\{w\} &= 0; & \text{Im}\{w\} &= 0 \\ f &= 3.2 \text{ kHz}; & \text{Re}\{w\} &= \frac{1}{2}\pi \text{ (increasing)}; \\ & & \text{Im}\{w\} &= \pi \text{ (increasing)}; \end{aligned}$$

Fig. 3.9 (b) (0,2) mode;

$$\begin{aligned} f &= 8.06 \text{ Hz}; & \text{Re}\{w\} &= \pi; & \text{Im}\{w\} &= 0 \\ f &= 3.2 \text{ kHz}; & \text{Re}\{w\} &= \frac{1}{2}\pi; & \text{Im}\{w\} &= 0 \end{aligned}$$

No problems were encountered with convergence which ran smoothly and rapidly in both cases. Using Newton-Raphson's method, 858 iterations were used to produce Fig. 3.9 (a) and 4171 iterations were used to produce Fig. 3.9 (b) with Muller's method. In both cases the attenuation was calculated for 397 evenly spaced frequencies in the interval 0 Hz - 3.2 kHz, hence the frequency resolution of 8.06 Hz. It seems that again it is the (0,2) mode that is the least attenuated above cut on.

Finally, another example of this 'anomalous behaviour' is taken from Wassilieff, Fig. 4. (a) in [ref. 9] reproduced here as Fig. 3.10 (a) for comparison with Fig. 3.10 (b).

Wassilieff maintains that the first part of the attenuation curve (to the left of the peak) is due to the (0,0) mode and that the remainder (to the right of the peak) is due to the (0,2) mode. The author has not yet found in any of the published literature attenuation curves that show the modes separately, such as those presented in this thesis.

Applying the same techniques as those employed for the parameter study in this subsection the author has been able to separate the (0,0) mode from the (0,2) mode and thereby verify the observation made by Wassilieff, compare figures (a) and (b).

Fig. 3.10 (a) is shown in more detail in Fig. 3.11 (a) and (b). Note though the logarithmic scale and compare with Fig. 3.9.

Please note that the dots in Fig. 3.10 (a) are measurements. In view of the excellent agreement between figures (a) and (b) these measurements serve also to verify the algorithm 'Extended'. The author's own verification is the subject of the next section, § 4.

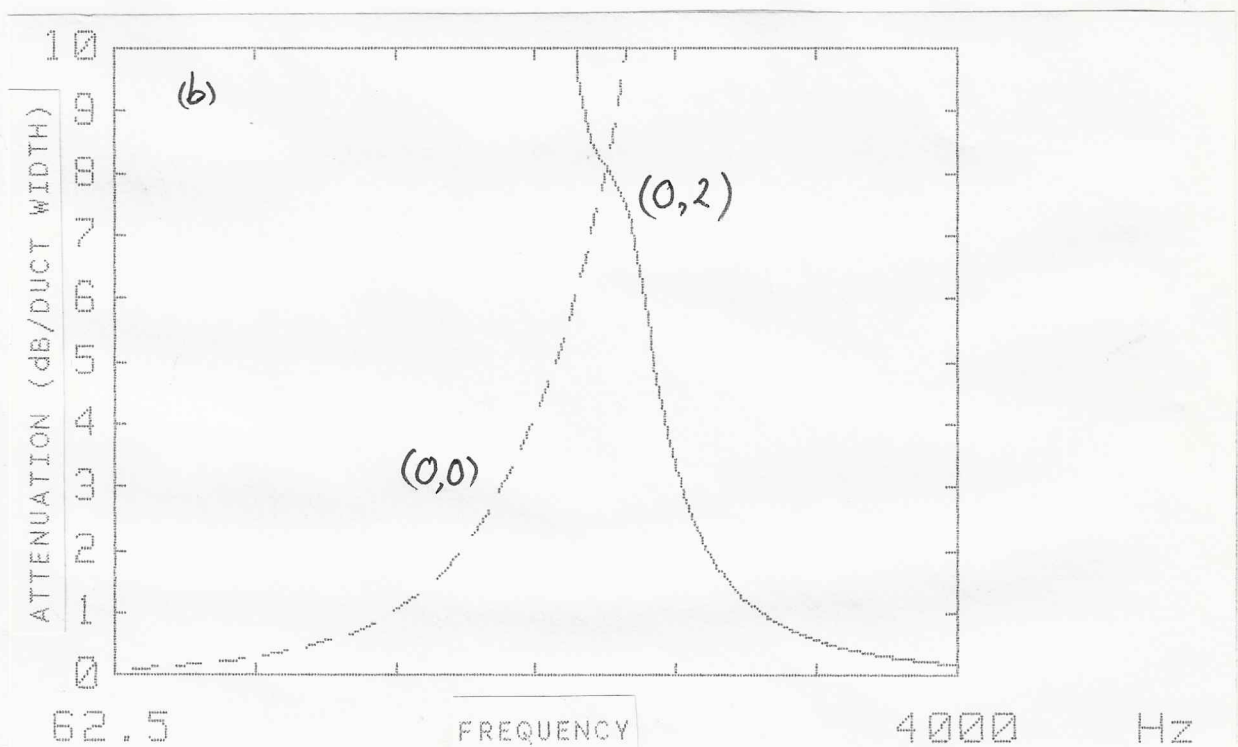
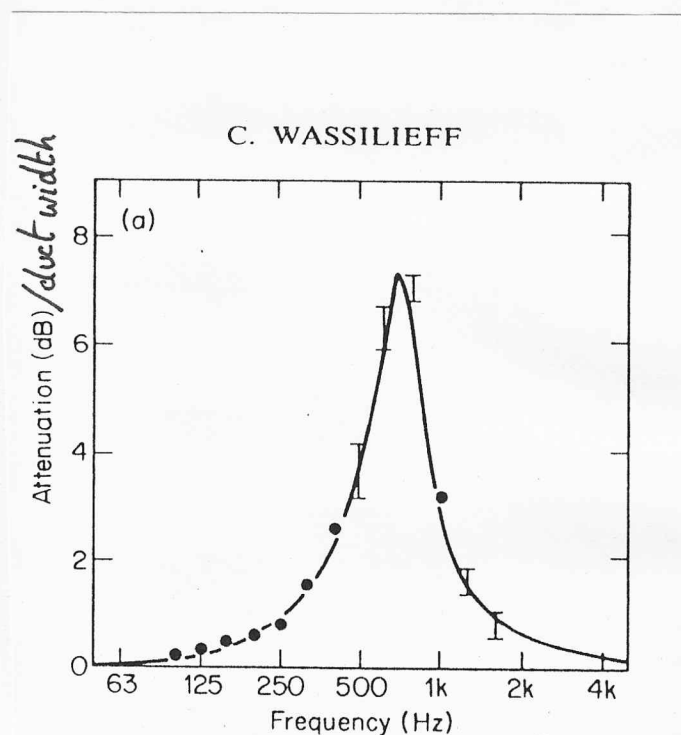


Fig. 3.10. (a) Reproduced from Fig. 4. (a) [ref. 9].
 (b) Produced using 'Extended'
 Γ attenuation; w Nyquist plot
 Freq. res. 20.0 Hz; $\sigma = 8,500 \text{ N}\cdot\text{s/m}$.
 Duct width $l = 0.5 \text{ m}$; lining thickness $d = 0.05 \text{ m}$

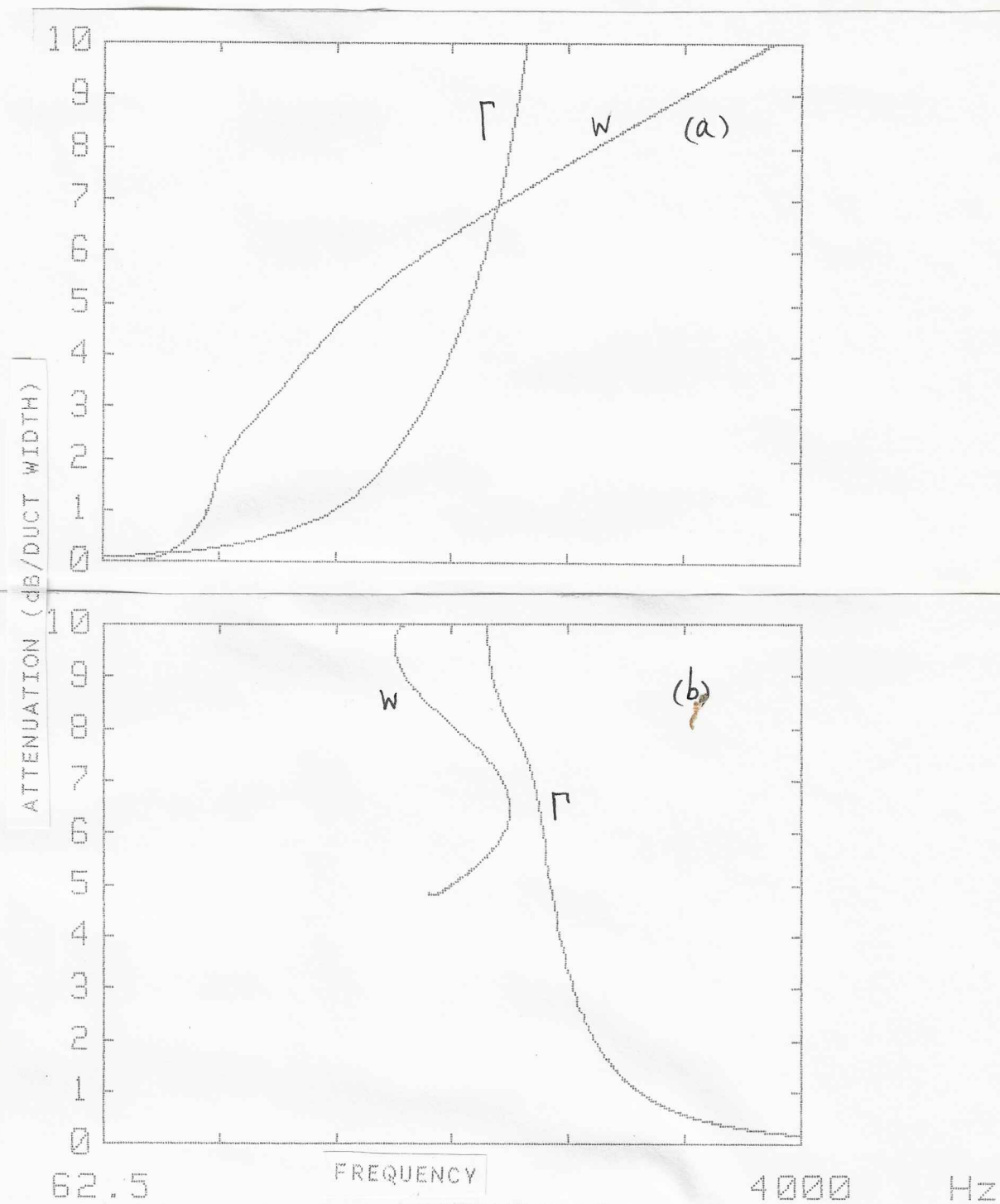


Fig. 3.11. (a) mode (0,0); (b) mode (0,2)
 Γ attenuation; w Nyquist plot
 Freq. res. 20.0 Hz; $\sigma = 8,500 \text{ N}\cdot\text{s/m}$.
 Duct width $l = 0.5 \text{ m}$; lining thickness $d = 0.05 \text{ m}$

3.5. Programme printouts 'Material' and 'Extended'

```

10 ! MARTIN and MARIA--USES A COMBINATION OF NEWTON-RAPHSON AND ANOTHER
    ! ITERATIVE METHOD FOR CALCULATING EXT. REAC.
20 !
30 ! PROGRAM FOR COMPUTING TRANSMISSION PROPERTIES OF DUCTS
40 !
50 ! Michael Grant paa den Hellige Tre Kongers Fest (Epiphany) Jan. 8 1989
60 !
70 ! PRINTER IS 1
80 ! ALPHA ON
90 ! GINIT
100 ! PLOTTER IS 3,"INTERNAL"
110 ! GCLEAR
120 ! OPTION BASE 1
130 !
140 ! PRINT CHR$(12)
150 ! PRINT "
160 ! PRINT " *****
170 ! PRINT " ***** MARTIN & MARIA *****
180 !
190 ! THE FOLLOWING IS A LIST OF THE IMPORTANT SYMBOLS USED IN THIS PROGRAM
200 !
210 !
220 ! A      NOMALISED FLOW RESISTANCE OF A ONE WAVELENGTH DEEP LAYER.
230 !      (See Acoustics of Ducts and Mufflers [M.L.Munjall] p.139.)
240 ! Ar, Ai, Br, Bi, Cr, Ci, Dr, Di, ARE THE Re. AND Im. MATRIX ELEMENTS
250 ! Air    SOURCE IMP. AS A PERCENTAGE OF CHAR. IMP. OF AIR
260 ! C      THE FREE SPACE SPEED OF SOUND
270 ! Damp    INSERTION LOSS
280 ! Fres    SPECTRAL RESOLUTION (NUMBER OF SPEC.COMP./MAX. FREQ.[Fmax])
290 ! Gam      RATIO OF SPECIFIC HEATS FOR AIR (1.41)
300 ! Gami, Gamr ARE THE Im. AND Re. PROP. CONSTANT OF LINING MATERIAL
310 ! Gi AND Gr " " " " " " " " " " DUCT
320 ! H, Hi, Ho ARE PIPE HEIGHTS WHERE i AND o REFER TO INNER AND OUTER
330 ! I      ACCUMULATIVE NUMBER OF ELEMENTS (FINALLY THE TOTAL NUMBER)
340 ! Imp     CHAR. IMP. OF AIR
350 ! J      NUMBER OF "INTERVAL" ALONG THE FREQUENCY AXIS
360 ! Kf      "NORMALISED" WAVE NUMBER (Kf=2*PI*Fres/C)
370 ! L      LENGTH OF ELEMENT
380 ! Ltot    ACCUMULATIVE LENGTH OF ELEMENTS (FINALLY IS TOTAL LENGTH)
390 ! M      MACH NUMBER
400 ! Max     NUMBER OF SPECTRAL COMPONENTS
410 ! N      NUMBER OF CURRENT ELEMENT BEING INVESTIGATED
420 ! Ny      COEFFICIENT OF DYNAMIC VISCOSITY
430 ! Nyc     IS RELATED TO DYNAMIC VISCOSITY Ny (Nyc=SQR(Ny)/C)
440 ! Ps      AMBIENT PRESSURE
450 ! R, Ri, Ro ARE PIPE RADII (Ri IS ALSO USED FOR LINING THICKNESS)
460 ! THE INDICES i AND o REFER TO INNER AND OUTER
470 ! Rho     DENSITY OF AIR
480 ! Sigma    FLOW RESISTANCE OF LINING MATERIAL
490 ! Te      TEMPERATURE IN CELCIUS
500 ! Vf      FLOW SPEED
510 ! W, Wo, Wi, ARE PIPE WIDTHS WHERE i AND o REFER TO INNER AND OUTER
520 ! X      THIS CAN EQUALL EITHER R OR H
530 ! Zd      CHAR. IMP. OF PIPE WITH NON-ABSORBANT WALLS
540 ! Zi, Zr  ARE THE Im. AND Re. SPEC. ACOUSTIC IMP. OF LINING MATERIAL
550 !         DIVIDED BY RHO*C
560 ! Zrad     RADIATION IMPEDANCE (EQUALS Zd FOR ANECHOIC TERMINATION)
570 ! Zs      SOURCE IMP. IN TERMS OF Zd
580 !
590 ! COM /Data/ REAL Zd(16),Zs,L(16),Ltot,R(16),Ri(16),Ro(16),W(16),Wo(16),Wi(1
600 ! 6),H(16),Ho(16),Hi(16),X(16),Af,Kf,Nyc,Fres,Gam,M,Rho
610 ! COM /Tally/ INTEGER I,J,N,Max
620 ! COM /Abs/ REAL A(400),Zr(400),Zi(400),Gamr(400),Gami(400),Gr(400),Gi(400),
630 ! Wre(400),Wim(400),T(400)
640 ! COM /Four/ REAL Ar(16,400),Ai(16,400),Br(16,400),Bi(16,400),Cr(16,400),Ci(
650 ! 16,400),Dr(16,400),Di(16,400)
660 ! COM /Rad/ REAL Zrad(400),Damp(400)

```



```

7910 !                               Material (properties of absorbing material)
7920 SUB Material
7930 !
7940 !   THE following is based on the empirical formulae of Delany and Bazley
7950 !   as modified by F. P. Mechel: ACOUSTICA vol.35 (1976) p. 210 - 213.
7960 !
7970 COM /Data/ REAL Zd(*),Zs,L(*),Ltot,R(*),Ri(*),Ro(*),W(*),Wo(*),Wi(*),H(*
),Ho(*),Hi(*),X(*),Af,Kf,Nyc,Fres,Gam,M,Rho
7980 COM /Tally/ INTEGER I,J,N,Max
7990 COM /Abs/ REAL A(*),Zr(*),Zi(*),Gamr(*),Gami(*),Gr(*),Gi(*),Wre(*),Wim(*
),T(*)
8000 !
8010 DISP "COMPUTING PROPAGATION CONSTANT OF ABSORBING MATERIAL"
8020 !
8030 K1=-1.4664
8040 K2=.2115
8050 K4=1.4035
8060 FOR J=1 TO Max
8070   A(J)=Af/J           ! A NORMALISED FLOW RESISTANCE, SEE REF.
8080   K3=A(J)/2/PI
8090   K5=K2*A(J)
8100   K=J*Kf
8110   IF A(J)<60 THEN
8120     Log=LOG(A(J))
8130     Zr(J)=1+.04885*EXP(.754*Log)      ! LINING CHARAC. IMPEDANCE
8140     Zi(J)=-.087*EXP(.7307*Log)
8150     Gamr(J)=K*.189*EXP(.6185*Log)     ! LINING PROPAGATION CONST.
8160     Gami(J)=K*(1+.0978*EXP(.6929*Log))
8170   ELSE
8180 !
8190     CALL Csqn(K1,K5,D1,D2)
8200     CALL Cdiv(K3,K4,D1,D2,E1,E2)
8210 !
8220     Zr(J)=E1           ! LINING CHARAC. IMPEDANCE
8230     Zi(J)=E2
8240     Gamr(J)=K*D1       ! LINING PROPAGATION CONST.
8250     Gami(J)=K*D2
8260   END IF
8270 NEXT J
8280 SUBEND

```

```

8870 !                                     Extended (material of extended reaction)
8880 SUB Extended(Fr$,S$)
8890 !
8900 COM /Data/ REAL Zd(*),Zs,L(*),Ltot,R(*),Ri(*),Ro(*),W(*),Wo(*),Wi(*),H(*
),Ho(*),Hi(*),X(*),Af,Kf,Nyc,Fres,Gam,M,Rho
8910 COM /Abs/ REAL A(*),Zr(*),Zi(*),Gamr(*),Gami(*),Gr(*),Gi(*),Wre(*),Wim(*
),T(*)
8920 COM /Tally/ INTEGER I,J,N,Max
8930 !
8940 DISP "COMPUTING PROPAGATION CONSTANT OF LINED DUCT (MATERIAL OF EXTENDED
REACTION)"
8950 !
8960 B=2*Ri(N)/X(N) ! b IN SCOTT/CHRISTIE EQUATION
8970 L1=X(N)/2 ! half the duct width (air space)
8980 K3=1 ! must be 1 or another +/- integer, not zero
8990 K4=0 ! this is a flag
9000 !
9010 INPUT "WAVE TYPE (0,0) MODE (0) OR (0,2) MODE (1) (default (0)) ?:",K4
9020 IF K4<>0 AND K4<>1 THEN
9030 BEEP 5000,.1
9040 GOTO 9010
9050 END IF
9060 DISP "COMPUTING PROPAGATION CONSTANT OF LINED DUCT (MATERIAL OF EXTENDED
REACTION)"
9070 K5=0 ! a flag
9080 Ep=1.0E-15 ! ERROR BOUND
9090 D5=0
9100 D6=0
9110 !
9120 FOR J=1 TO Max
9130 Gr(J)=D5
9140 Gi(J)=D6 ! START VALUES, DUCT PROPAGATION CONST.
9150 K=Kf*J ! wave number
9160 !
9170 CALL Cmult(Gamr(J),Gami(J),Gamr(J),Gami(J),D1,D2)!Gam*Gam (material)
9180 !
9190 CALL Cmult(Gr(J),Gi(J),Gr(J),Gi(J),D3,D4) ! G*G (duct)
9200 !
9210 CALL Cmult(Zr(J),Zi(J),Gamr(J),Gami(J),E1,E2) ! (Z/rho/C)*Gam
9220 !
9230 CALL Cdiv(0,K,E1,E2,M1,M2) ! jK*rho*C/(Z*Gam) M1=Re M; M2=Im M
9240 !
9250 W3=(D3+(K*K))*L1*L1 ! W3=Re w*w
9260 !
9270 W4=(D4*L1*L1) ! W4=Im w*w
9280 !
9290 P3=-(D1+(K*K))*L1*L1 ! P3=Re p*p
9300 !
9310 P4=-(D2*L1*L1) ! P4=Im p*p
9320 !
9330 CALL Csqr(W3+P3,W4+P4,X1,X2) ! X1=Re SQR(w*w+p*p); X2=Im SQR(w*w+p*p)
9340 !
9350 CALL Ctan(B*X1,B*X2,X3,X4) ! TAN(b*SQR(w*w+p*p))
9360 !
9370 CALL Cmult(X1,X2,X3,X4,X5,X6) ! (SQR(w*w+p*p))*TAN(b*SQR(w*w+p*p))
9380 !
9390 CALL Cmult(M1,M2,X5,X6,X7,X8) ! M*(SQR(w*w+p*p))*TAN(b*SQR(w*w+p*p))
9400 !
9410 CALL Csqr(W3,W4,W1,W2) ! w
9420 !
9430 IF K4=1 THEN GOTO 10290
9440 !
9450 CALL Ctan(W1,W2,W5,W6) ! TAN(w) W5=Re TAN w; W6=Im TAN w
9460 !
9470 CALL Cmult(W1,W2,W5,W6,W7,W8) ! w*TAN(w)
9480 !
9490 F1=W7+X7 ! F1=Re f(w)

```



```

9500 !
9510 F2=W8+X8 ! F2=Im f(w)
9520 !
9530 CALL Csin(2*B*X1,2*B*X2,S1,S2) ! SIN(2*b*(SQR(w*w+p*p)))
9540 !
9550 CALL Ccos(2*B*X1,2*B*X2,C1,C2) ! COS(2*b*(SQR(w*w+p*p)))
9560 !
9570 A1=(2*B*X1)+S1
9580 ! 2*b*SQR(w*w+p*p)+SIN(2*b*(SQR(w*w+p*p)))
9590 A2=(2*B*X2)+S2
9600 !
9610 CALL Cdiv(A1,A2,C1+1,C2,A3,A4)
9620 !
9630 CALL Cmult(A3,A4,W1,W2,A5,A6)
9640 !
9650 CALL Cdiv(A5,A6,X1,X2,A7,A8)
9660 !
9670 CALL Cmult(M1,M2,A7,A8,B1,B2)
9680 !
9690 CALL Csin(2*W1,2*W2,S3,S4) ! SIN(2*w)
9700 !
9710 CALL Ccos(2*W1,2*W2,C3,C4) ! COS(2*w)
9720 !
9730 B3=2*W1+S3
9740 !
9750 B4=2*W2+S4
9760 !
9770 CALL Cdiv(B3,B4,C3+1,C4,B5,B6)
9780 !
9790 F3=B1+B5 ! F3=Re f'(w)
9800 !
9810 F4=B2+B6 ! F4=Im f'(w)
9820 !
9830 CALL Cdiv(F1,F2,F3,F4,F5,F6)
9840 !
9850 CALL Cdiv(F5,F6,W1,W2,F7,F8)
9860 !
9870 IF ABS(F7)>Ep OR ABS(F8)>Ep THEN GOTO 9910 ! COMPARE WITH Ep
9880 !
9890 IF ABS(F7)<=Ep AND ABS(F8)<=Ep THEN GOTO 10030
9900 !
9910 W1=W1-F5
9920 !
9930 W2=W2-F6
9940 !
9950 CALL Cmult(W1,W2,W1,W2,W3,W4)
9960 !
9970 T(J)=T(J)+1
9980 !
9990 IF T(J)>100 THEN GOTO 10030
10000 !
10010 GOTO 9330
10020 !
10030 Wre(J)=W1
10040 !
10050 Wim(J)=W2
10060 !
10070 IF J>=3 THEN GOTO 10110
10080 !
10090 IF J<3 THEN GOTO 10840
10100 !
10110 E1=ABS(Wre(J-1)-Wre(J-2))
10120 !
10130 E2=ABS(Wim(J-1)-Wim(J-2))
10140 !
10150 E3=ABS(Wre(J)-Wre(J-1))
10160 !
10170 E4=ABS(Wim(J)-Wim(J-1)) ! LINE 10190: COMPARES VECTOR LENGTHS
10180 !
10190 IF SQR((E1*E1)+(E2*E2))>SQR((E3*E3)+(E4*E4))/3 THEN GOTO 10840
10200 !
10210 K5=K5+1 ! THE PRESENT METHOD HAS FAILED. A "JUMP" HAS OCCURRED

```

```

10220!
10230      IF K5=2 THEN GOTO 10670      ! IF K5=2 WHEN BOTH METHODS FAIL
10240      K4=ABS(K4-1)                ! This is a flag
10250      T(J)=0                      ! J=J-2
10260      T(Max+2)=T(Max+2)+1
10270      GOTO 9150
10280!
10290      CALL Cdiv(-X7,-X8,W1,W2,Y1,Y2)      !
10300!
10310      J1=SQR((Y1*Y1)+((Y2+1)*(Y2+1)))
10320!
10330      J3=SQR((Y1*Y1)+((Y2-1)*(Y2-1)))
10340!
10350      CALL Arctan(Y1,(Y2+1),Phi)
10360!
10370      J2=Phi
10380!
10390      CALL Arctan(-Y1,(1-Y2),Phi)
10400!
10410      J4=Phi
10420!
10430      Y3=(K3*PI)+((J4-J2)/2)
10440!
10450      Y4=(1/2)*(LOG(J1/J3))
10460!
10470      En=1.0E-10
10480      IF ABS(W1-Y3)>En OR ABS(W2-Y4)>En THEN GOTO 10520
10490!
10500      IF ABS(W1-Y3)<=En OR ABS(W2-Y4)<=En THEN GOTO 10620
10510!
10520      W1=ABS(Y3)
10530!
10540      W2=Y4
10550!
10560      CALL Cmult(W1,W2,W1,W2,W3,W4)
10570!
10580      T(J)=T(J)-1
10590      IF T(J)<-100 THEN GOTO 10620
10600      GOTO 9330
10610!
10620      Wre(J)=W1
10630!
10640      Wim(J)=W2
10650!
10660      GOTO 10070
10670      K4=ABS(K4-1)
10680      K6=N
10690      FOR N=J TO J+1
10700          Wre(N)=2*Wre(N-1)-Wre(N-2)      ! LINEAR EXTRAPOLATION
10710          Wim(N)=2*Wim(N-1)-Wim(N-2)
10720          W1=Wre(N)
10730          W2=Wim(N)
10740          CALL Cmult(W1,W2,W1,W2,W3,W4)
10750          CALL Csqr((W3/L1/L1)-(K*K),W4/L1/L1,D5,D6)
10760          Gr(N)=D5
10770          Gi(N)=D6                          !Gi AND Gr, DUCT PROPAGATION CONST.
10780          Gr(N+1)=Gr(N)
10790          Gi(N+1)=Gi(N)
10800      NEXT N
10810      N=K6
10820      J=J+1
10830      GOTO 10940
10840      CALL Csqr((W3/L1/L1)-(K*K),W4/L1/L1,D5,D6)
10850!
10860      Gr(J)=ABS(D5)
10870!
10880      Gi(J)=ABS(D6)                          ! DUCT PROPAGATION CONSTANT
10890!
10900      IF J<2 THEN GOTO 10940
10910      D5=ABS(2*Gr(J)-Gr(J-1))                ! LINEAR EXTRAPOLATION
10920      D6=ABS(2*Gi(J)-Gi(J-1))
10930      !

```

```

10940     K5=0
10950     NEXT J
10960     K6=N
10970     FOR N=1 TO Max
10980         T(Max+3)=T(Max+3)+ABS(T(N))
10990         IF Fr$="2" THEN          ! COR. FAC. 2 FOR Sq. DUCTS LINED ALL AROUND
11000             Gr(N)=2*Gr(N)
11010         END IF
11020         IF S$="C" THEN          ! CORRECTION OF A FACTOR OF 2 FOR CIRC. DUCTS
11030             Gr(N)=2*Gr(N)
11040         END IF
11050     NEXT N
11060     N=K6
11070 SUBEND

```


4. MEASUREMENTS

4.1. Insertion loss

In order to help verify the algorithm 'Extended', insertion loss measurements were performed on the silencer which is shown in the photographs in Fig. 4.1 and drawn schematically in Fig. 4.2.

Measurements were performed in accordance with the following definition of insertion loss (IL) from [ref. 14 §; 2.10.1]:

Insertion loss is defined as the difference between the acoustic power without any filter and that with the filter.

One can conceive of two alternative reference systems:

- i) a duct with the same length and cross section as the silencer system,
- ii) an infinitely short duct, i.e. radiation from an orifice with the same cross section as the duct.

It is implicit in the above that the exhaust (or inlet) pipe, the tail pipe and the reference duct must have the same cross sectional dimensions. This is because the radiation impedance must be the same in each case, any difference in sound power output must be due only to the filter. See also the discussion in [ref. 8; §§ 2.3 & 3.2].

A small loud speaker was used as the sound source, this is far from being a simple piston source and so for practical reasons the IL was measured according to alternative i) above. This ensured that the radiation impedance remained constant throughout.

Although measurements were performed in accordance with the above definition the procedure was modified slightly and the intensity at a fixed point was measured rather than the sound power. A measurement of the sound power would require that the intensity be integrated over a surface bounding on the duct orifice (this simply follows from the integral below).

$$P = \int_{\text{surface}} I(s) \cdot ds$$

For reliable comparisons to be made between measurements and theory the flow resistance of the lining material was required. For reasons given in § 4.2 this could only be measured for frequencies up to 3.2 kHz and then only reliably up to 1.6 kHz, see [ref. 20]. Measurements of the IL were therefore only made for frequencies up to 3.2 kHz.

A spacing of 12 mm was chosen for the intensity probe. This spacing has an associated error no greater than -1 dB in the frequency range 100 Hz to 4 kHz (see discussion below and also [ref. 21]). The centre of the intensity probe was placed 10 cm



Fig. 4.1 (a) The experimental set up.

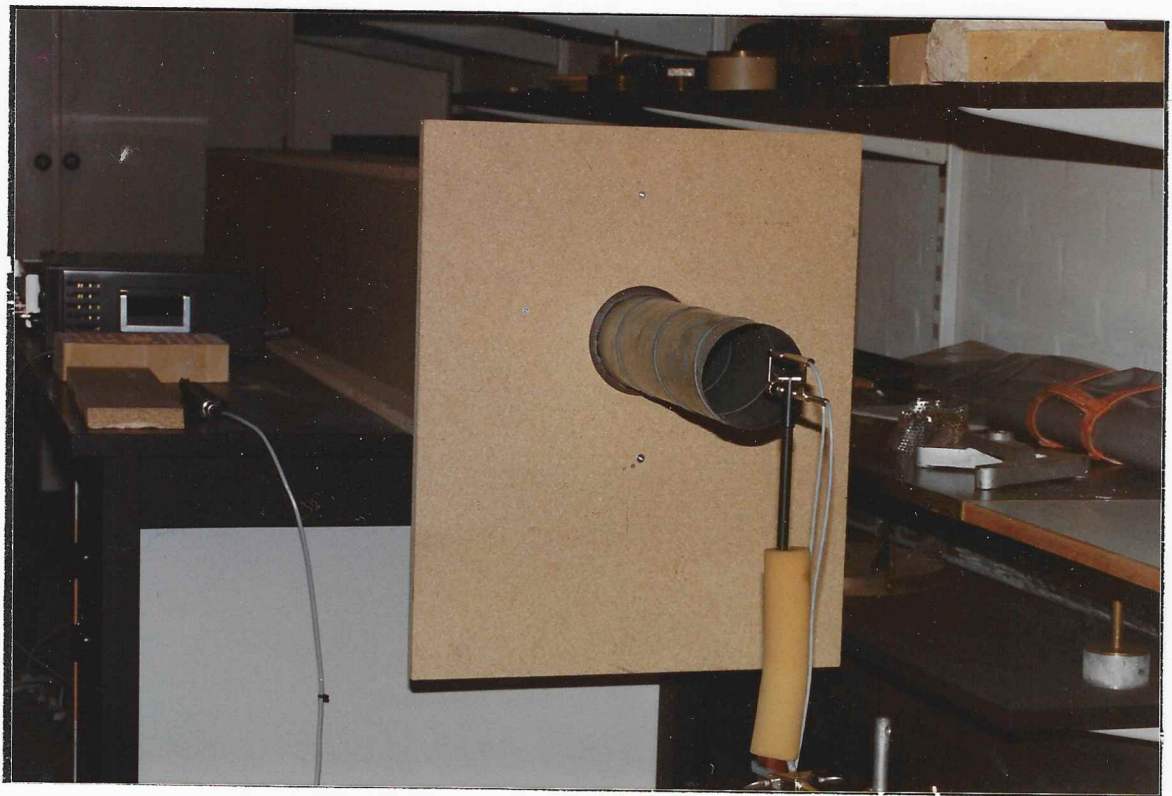


Fig. 4.1 (b) Positioning the intensity probe.

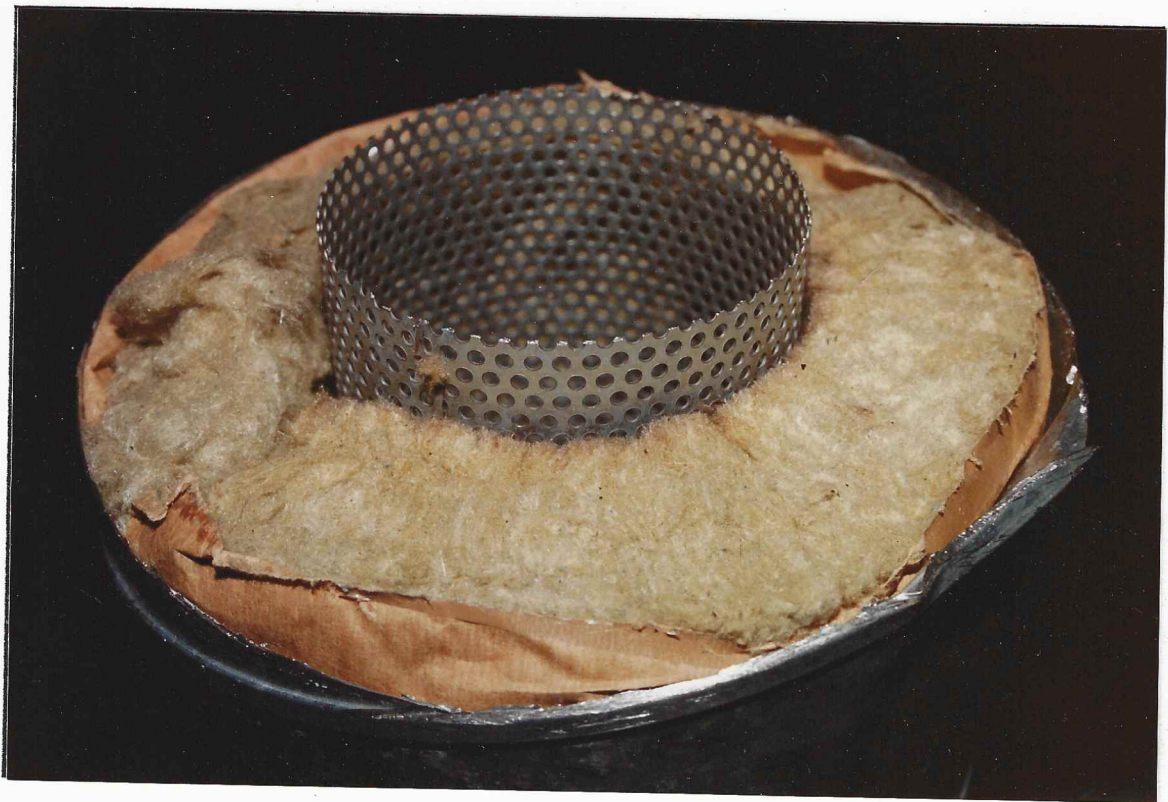


Fig. 4.1 (c) The silencer. Notice the structure of the lining.

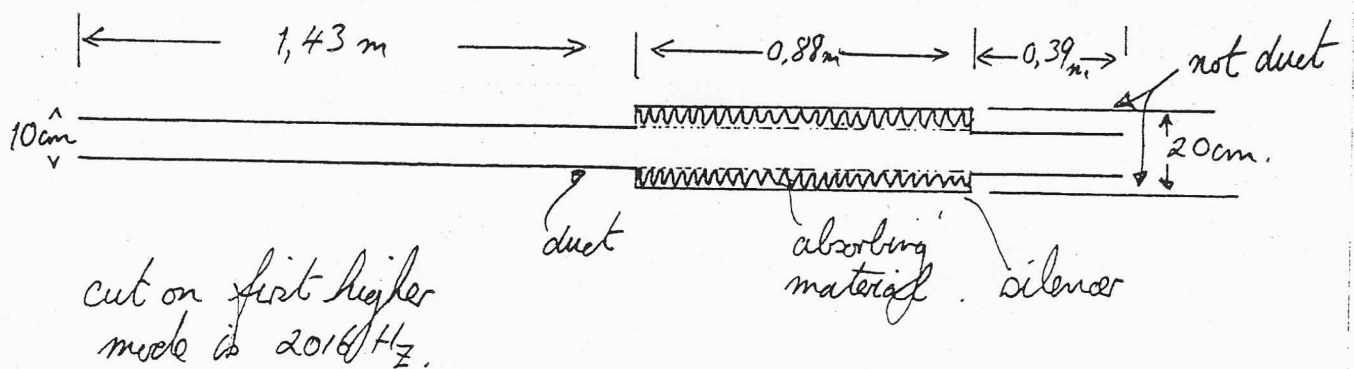


Fig. 4.2 The silencer in place in the duct (schematic).

from the duct orifice and on and parallel to the duct axis. This position was maintained throughout, having been decided upon after considerations (outlined below) of the sound field in the vicinity of the duct orifice.

For simplicity's sake the duct orifice can be represented as a rigid vibrating piston. This is of course an idealisation since not all points in the plane of the duct orifice move in phase and with the same amplitude. A further idealisation that allows calculations of the beam pattern to be made is to consider the piston to be mounted flush with an infinite baffle. With the above idealisations one can write,

$$\frac{2 \cdot J_1(kr \sin \theta)}{kr \sin \theta} = 10^{-m/20},$$

where θ is that angle off the duct axis (measured from the centre of the duct orifice) for which the intensity will have decreased m dB.

With a maximum frequency of 3.2 kHz and a duct radius of 5 cm the maximum value for kr is 2.92 and on setting respectively $m = 1$ and $m = 3$, one obtains by iteration (or tables) that $\theta = 19.0^\circ$ and 33.5° . When $\theta = 90^\circ$ it is found that the intensity is down by 12 dB (i.e. $m = 12$) from the value on the axis and that the first side lobes appear for frequencies greater than 4.2 kHz. Fresnel zones first appear for $kr < \pi$, (in this case $f > 3.44$ kHz) and the first higher order mode in the duct appears when $kr > 1.84$ (in this case given by $f_{1,1} = 2.02$ kHz).

The unflanged and flanged ducts radiate respectively into solid angles 4π and 2π . With this and the above approximations in mind the following can be concluded. The beam patterns for the frequency range of interest can be considered to be spherical. The uncertainty in repositioning the microphone between successive measurements on the duct, with and without the silencer, was much less than 19.0° and one can assume that the probe has not been placed between side lobes or in such a way that would create a poor signal/noise ratio. In addition, care was taken to place the probe further than $0.6 r$ ($10 \text{ cm} > 3 \text{ cm} = 0.6 r$) from the duct orifice. This distance is of course the end correction for an unflanged pipe. To place the probe within this distance or a little way inside the duct orifice with a view to improving the signal strength would be counter productive. The sound field in this region is reactive, sound energy is being reflected back down the duct and standing waves are present. To place the probe further away would likewise reduce the signal/noise ratio, through a little trial and error the above compromise distance of 10 cm was found. In theory one should obtain the least error due to the phase difference at the microphones if there is a maximum pressure gradient between them. This, one would expect to obtain where there is 'pressure release' that is in this case at $0.6 r$ from the duct orifice. Measurement problems remained though due to the acoustical behaviour of the room within which the measurements were performed. Further, a 'perfect' intensity probe would not have these problems even in a very reactive sound field, since

this would simply contribute zero to the measurement. This follows directly from Gauss's theorem when using the integral above over a 'closed' surface.

The total sound pressure in the frequency range 960 - 4,160 Hz was found to be

49 dB (re. 20 μ Pa) microphone A
and
47 dB (re. 20 μ Pa) microphone B.

However the total intensity in the same frequency interval was found to be

39.4 dB (re. 10⁻¹²W/m²).

This serves as a measure of how reactive the sound field is. As a rule of thumb, intensity measurements in sound fields where the sound pressure is 10 dB or more than the intensity are difficult to carry out. The above is an indication that the measurements were performed under far from ideal conditions.

These measurements were made with the duct (plus the silencer) enclosed in a sound proofing wooden box specially designed for this purpose, see Fig. 4.1. This considerably reduced the extraneous noise, from the loud speaker sound source and from the walls of the duct. The silencer was found to attenuate by up to 80 - 90 dB and this sound enclosure was necessary to improve the signal/noise ratio in the somewhat reverberant room. Given the facilities available at the time, this was the best measurement set up that could be achieved.

As can be seen in Fig. 4.1 the measurements were performed using a Brüel & Kjær type 2032 dual channel FFT analyser controlled by an HP personal computer. Measurements were recorded on a floppy disc for later analysis.

A Brüel & Kjær type 1049 sine/noise generator was used to produce white noise in the range 200 Hz - 20 kHz. This was band pass filtered from 2 kHz - 3.2 kHz. The noise signal was afterwards amplified before being sent to the loud speaker.

The auto spectrum of the voltage across the loud speaker terminals was measured and stored on the floppy disc for each intensity measurement. Assuming linearity this enabled the intensity measurements to be normalised and allowance could therefore be made for any possible change in the experimental setup between successive measurements. The filter characteristics could for example change with temperature or the control setting on the amplifier may have been altered. Inherent in this approach is the assumption that throughout the course of a single measurement (intensity followed by auto spectrum for the voltage) these factors remain constant.

The silencer was found to contain a cylindrical perforated plate that served to hold the sound absorbing material in place, see

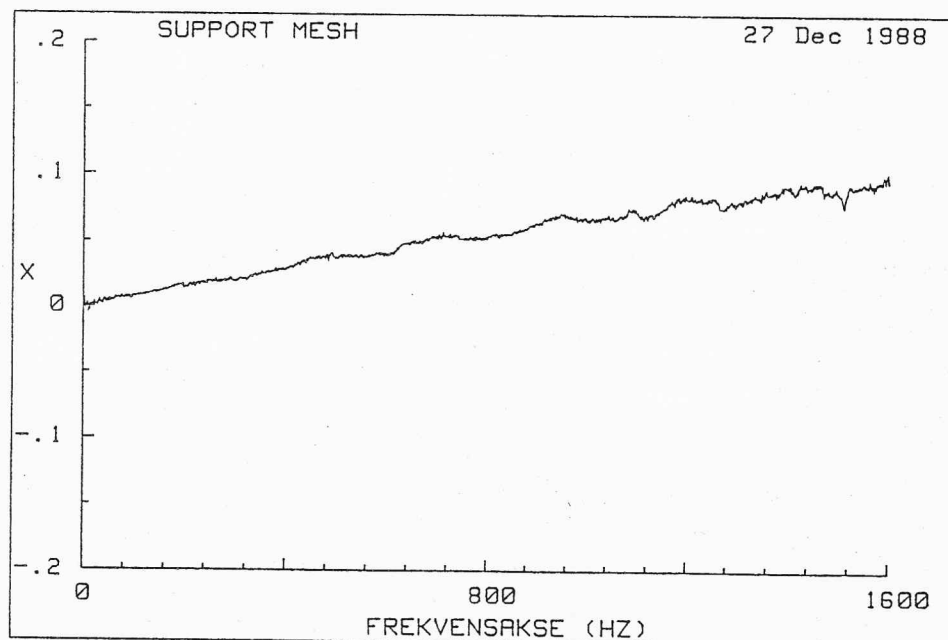
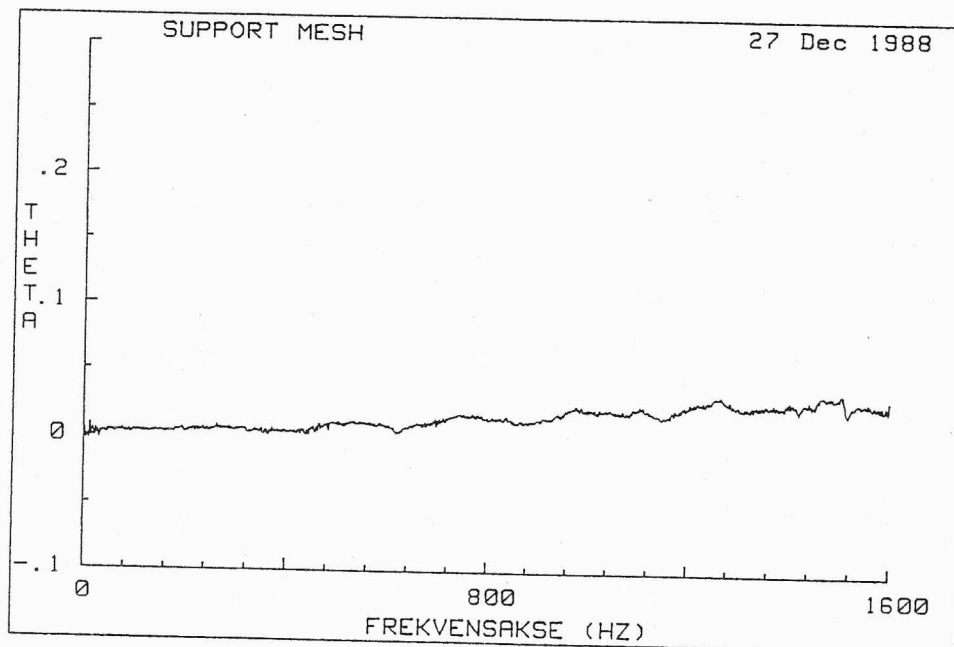


Fig. 4.3 The flow resistance of the perforated cylinder, used as a support mesh for the lining specimens. THETA is the real part of the flow resistance, X (Chi) is the imaginary part of the flow resistance.

again Fig. 4.1. Measurements on the perforated plate (for zero angle of incidence) showed that it had only a very small imaginary value of flow resistance the real part being even smaller. These values, as can be seen from Fig. 4.3 increase slowly with frequency. It should however be noticed that although the flow resistance is low the flow resistivity is quite large for higher frequencies. (Flow resistivity = flow resistance/thickness, and the plate is less than 1 mm in thickness.) The small real part of the flow resistance is due to viscous losses at the sides of the holes and the imaginary value is due to the potential energy stored in the mass of the small pockets of air in the holes (perforations) that oscillate in sympathy with the incident sound field.

In the Introduction (§ 1) the author anticipated that the presence of the perforated plate would reduce the attenuation at higher frequencies, something in fact contrary to what one would expect when calculating for local reaction. In this case the additional albeit very small flow resistance that the perforated plate contributes to the whole, would at the very worst not diminish the attenuation. Furthermore 'Extended' cannot calculate what effect the presence of the perforated plate has. It was therefore necessary to remove the plate.

The following three measurements were performed.

- i) the reference system (a straight duct of length 2.7 m with circular cross section of radius 5 cm
- ii) the system consisting of duct plus silencer with the cylindrical perforated plate
- iii) the system consisting of duct plus silencer without the cylindrical perforated plate.

The IL results from ii) and iii) above are shown in Fig. 4.4. It can be seen in Fig. 4.4 (a) which shows the result from iii) that the attenuation is in fact higher at higher frequencies when the perforated cylinder is not present. Notice that there is no difference at low frequencies between the two curves. This is also what was predicted by the author in the introduction to this thesis, see p. 5.

4.2. Flow resistivity

The flow resistivity has been calculated from measurements of the flow resistance made on the silencer material plus mesh. These measurements have been performed according to the method described in [ref. 20]. The mesh was taken from the perforated cylinder from inside the silencer. As already mentioned the flow resistance of this was found to be very small, negligible in fact (Fig. 4.3). A piece of the perforated plate was fashioned for use as a support in which to place a sample of the lining material. The sample was positioned in such a way that during the measurement the sound would be incident on it in the same manner

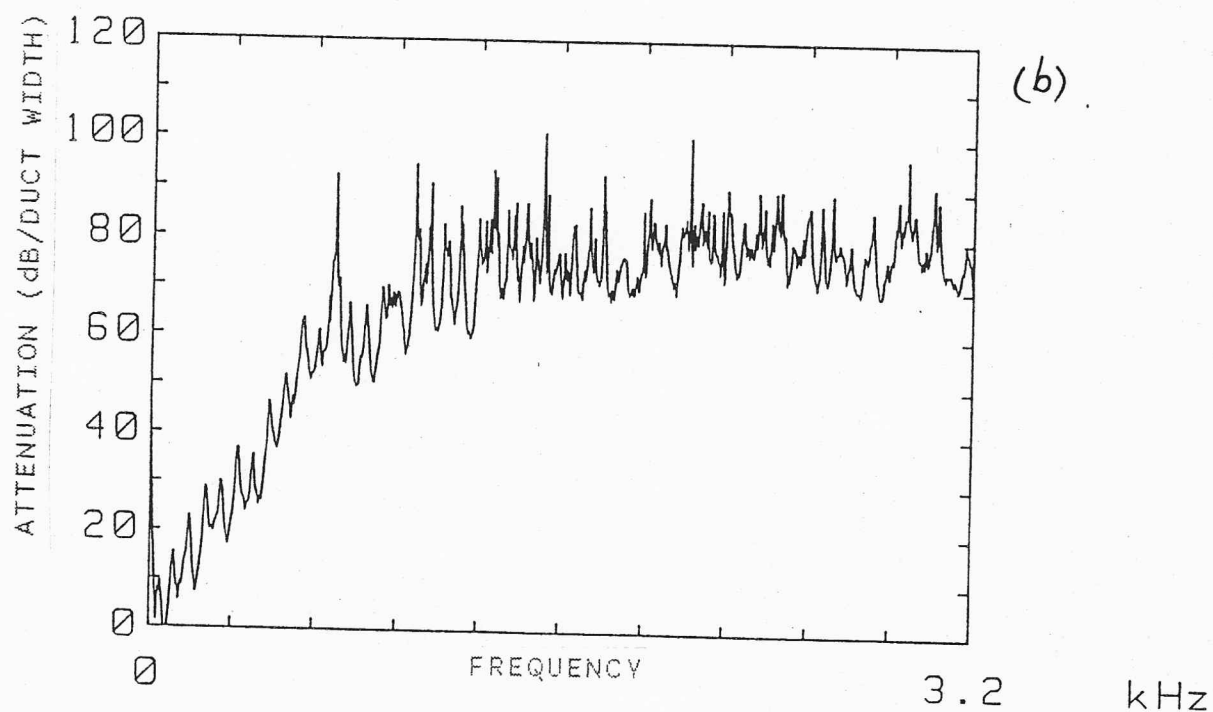
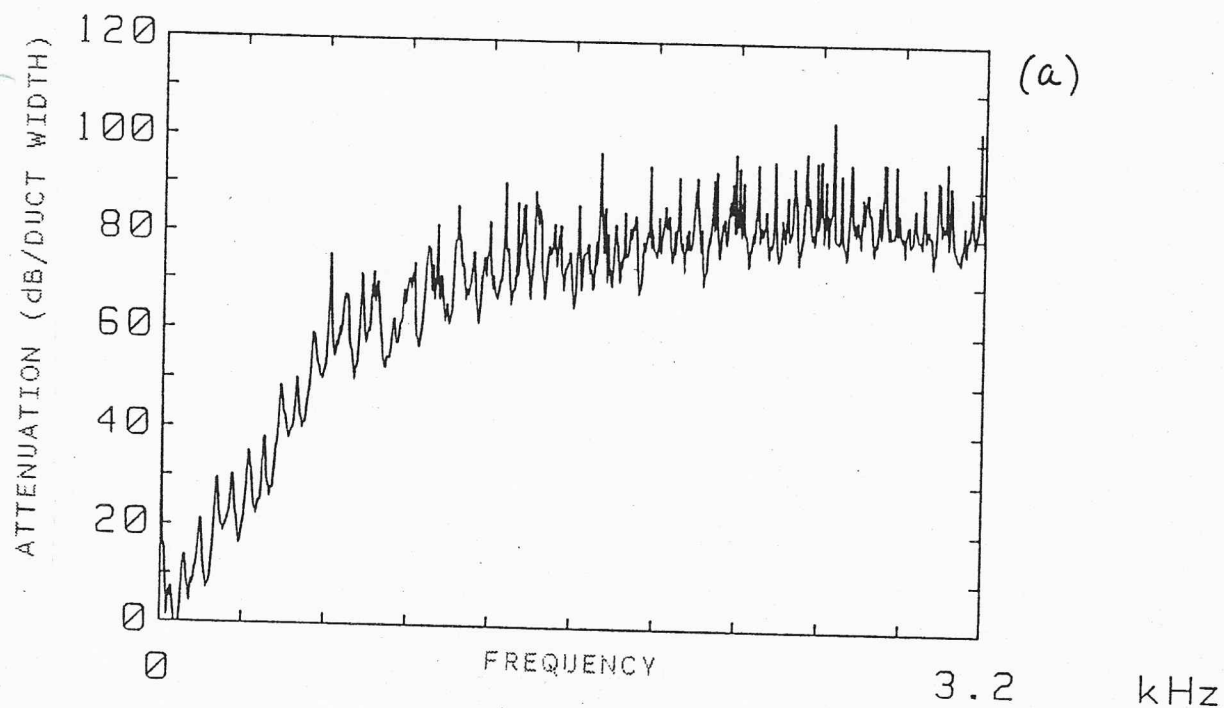


Fig. 4.4 The IL measured for the arrangement shown schematically in Fig. 4.2.
 (a) without the perforated cylinder.
 (b) with the perforated cylinder.

as if it were in the silencer. A close look at the photograph of the silencer in Fig. 4.1 shows that the lining material may be anisotropic, it was therefore very important to correctly position the sample. Results of the measurement are shown in Fig. 4.5. As is also the case in Fig. 4.3 the flow resistance is normalised to that of air (415 rayls). To obtain the flow resistivity one must divide by the sample thickness. The values of interest in this thesis are the real values, θ . The measurements are dynamic, there is a real and imaginary component to σ .

From Fig. 4.5 (a) one has for a sample thickness of 0.75 cm:

$$\text{at } f = 1600 \text{ Hz, } \theta = 0.6$$

$$\text{therefore } \sigma = (0.6 \cdot 415 / 0.0075) \text{ N}\cdot\text{s/m}^4 = 33,200 \text{ N}\cdot\text{s/m}^4.$$

The sample thickness was not uniform and therefore could only be measured to within an accuracy of ± 2 mm around 0.75 cm and θ was seen to increase for those measurements made up to around 2 kHz (approx. cut on freq. in the measuring duct for the first higher order mode). These two observations lead the author to conclude that the flow resistivity is better expressed as:

$$30000 \text{ N}\cdot\text{s/m}^4 < \sigma < 40000 \text{ N}\cdot\text{s/m}^4$$

4.3. Comparison with calculations

The algorithm 'Extended' was used to calculate the IL on the basis of the two values for σ given above and using the dimensions shown in Fig. 4.2. The results of these calculations which considered the radiation from the duct orifice are shown in Fig. 4.6. Although the measurements were performed on a duct with a circular cross section the algorithm attempted to compensate for this using eq. 2.1.3. See also lines 11020 to 11050 in 'Extended' (see p. 50).

The best fit between the calculated values and those measured is achieved if one starts to follow the attenuation curve in (b) ($\sigma = 40000 \text{ N}\cdot\text{s/m}^4$) and after cross over continues with curve (a) ($\sigma = 30000 \text{ N}\cdot\text{s/m}^4$). Part of the problem here is to reflect in the calculations the influence that frequency variation of σ will have on the results. This approach here is the only one available using 'Extended' where each calculation is performed throughout the whole frequency range with constant σ . Actually it was found that the results were not, in this case at least, that sensitive to marked variations in σ . See again Fig. 4.6.

The difference of 20 dB between calculations and measurement in part of the frequency range requires explanation.

Firstly the calculations with 'Extended' are in much better agreement with experiment than those obtained with Morse's theory. The theory developed by Scott assumes that the external duct wall is hard. That is far from the case in this experiment. The acoustic properties of the room were not particularly favourable

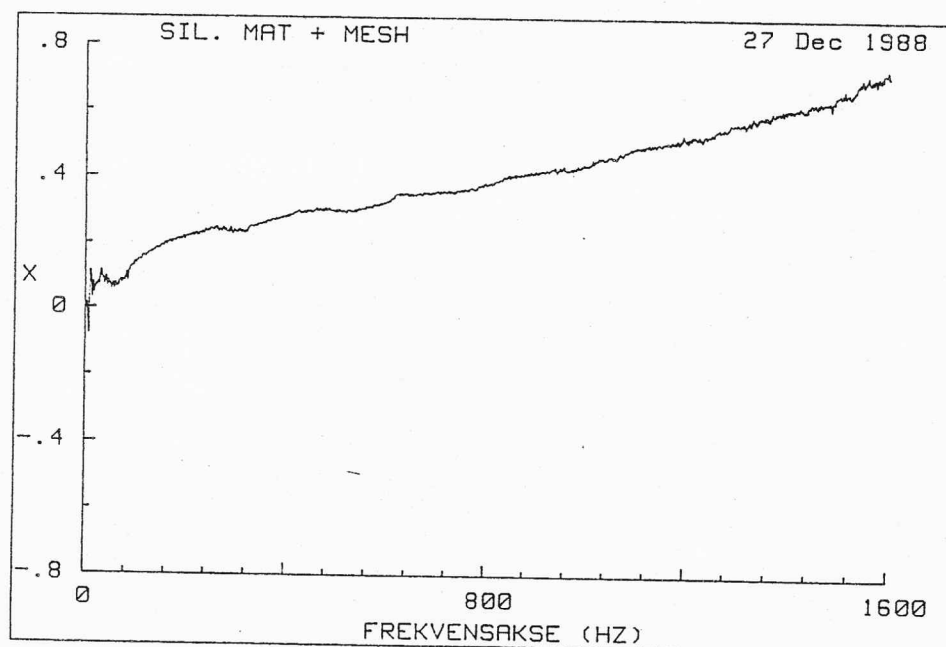
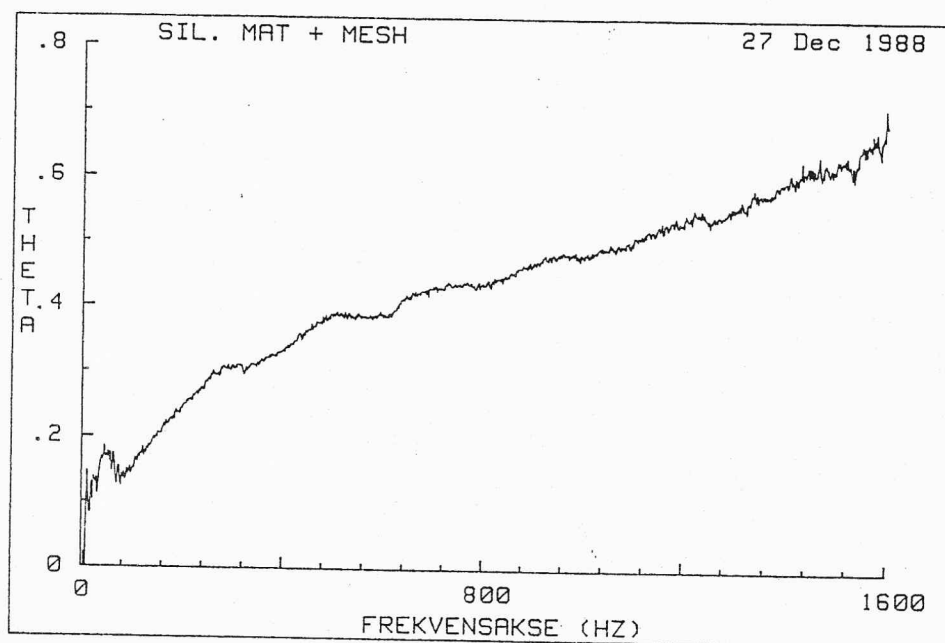


Fig. 4.5 The flow resistance of the lining together with the supporting mesh.
THETA is the real part of the flow resistance,
X (Chi) is the imaginary part of the flow resistance.

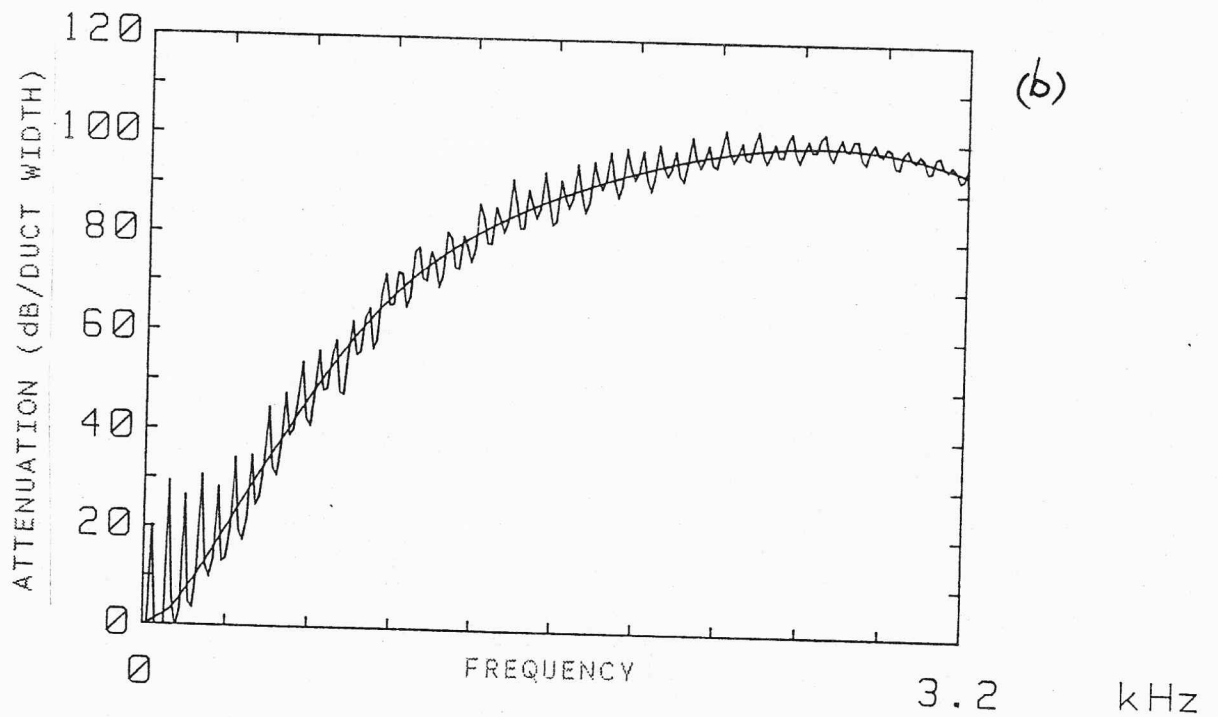
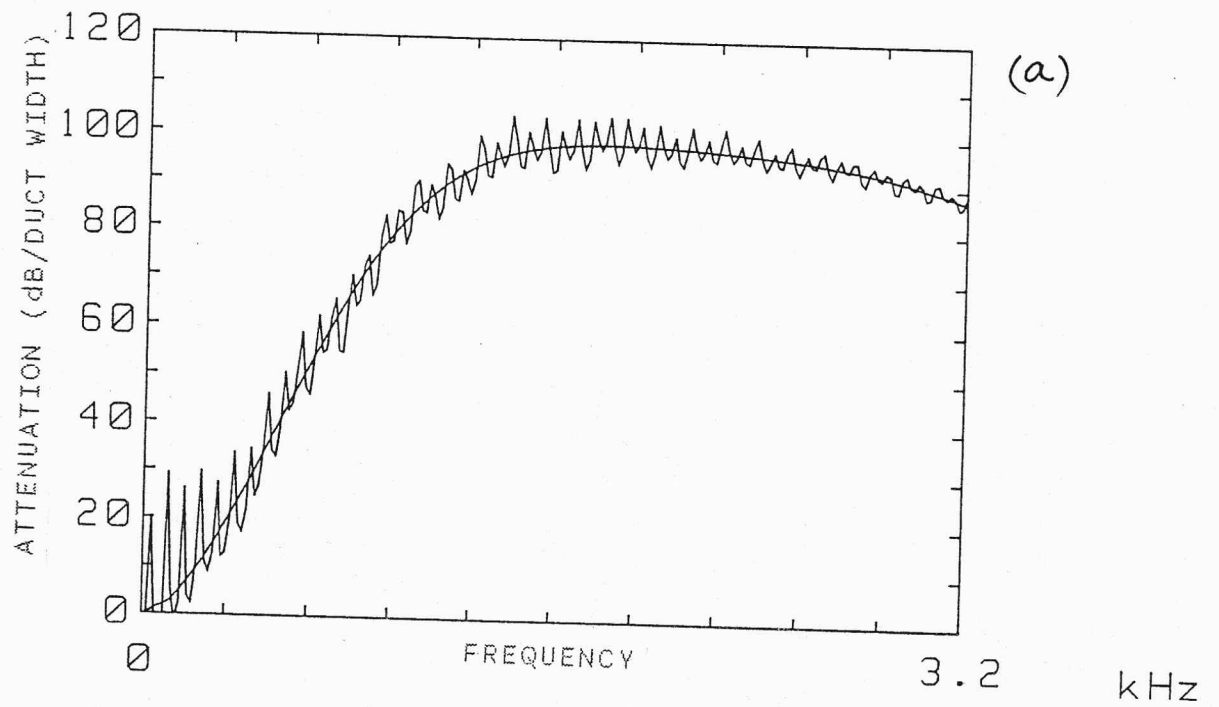


Fig. 4.6 IL calculated, using 'Extended', for the arrangement shown in Fig. 4.2.
 (a) $\sigma = 30000 \text{ N}\cdot\text{s}/\text{m}^4$; (b) $\sigma = 40000 \text{ N}\cdot\text{s}/\text{m}^4$

as has been noted above. Further these measurements are intended as a verification, complementary to that provided in the published literature, see again Fig. 3.10. However in this case the validity of using eq. 2.1.3 with 'Extended is also tested. It has been noticed that for rectangular ducts the (0,2) mode can be the least attenuated. That similar situations may arise with ducts with circular cross sections will put in doubt the application of eq. 2.1.3 as it is used above. Siding for caution the author can only conclude that more work has to be done, see § 5.

5. CONCLUSION

One of the most striking features revealed in the course of studies undertaken here by the author is the apparent anomalous behaviour of the $(0,0)$ and $(0,2)$ modes for frequencies above cut on for $(0,2)$. This behaviour occurs only for certain ratios of lining thickness/duct width and for certain flow resistivities. Note for instance the sudden change in behaviour between Fig. 3.3 (f) and Fig. 3.3 (h), and again for curve nos. 4 & 5 and 11 & 12 in Fig. 3.2.

A clue to understanding the physics of the process may be found in the quotation from Morse and Ingaard on p. 5. The idea of a separate $(0,0)$ and $(0,2)$ mode which is quite applicable for hard walled ducts is probably no longer fully applicable for soft walled ducts.

In the case of those $(0,0)$ modes that are not the least attenuated it is as if the sound energy is being ever more diffracted out towards the sides of the duct. How else would it be attenuated? This being so then the wave front that one has associated with $(0,0)$ is no longer plane. See also the comments concerning the author's way of picturing this process on p. 5.

It is probably safer therefore to say that the first series of solutions are associated with the $(0,0)$ mode in a hard walled duct and that the second series of solutions are those associated with the $(0,2)$ mode in a hard walled duct etc.

This terminology would perhaps be less confusing.

Calculations with 'Extended' such as those in § 4.3 for comparison with measured results are based on the assumption that it is the $(0,0)$ mode that is least attenuated. It is further assumed that therefore most of the energy is transmitted via this mode. This so it now seems is not necessarily the case. It would be useful therefore to also know how the sound energy is distributed between the two modes $(0,0)$ and $(0,2)$. It is possible that this information would allow a more accurate determination of the IL for the silencer in § 4. Scott's analysis excludes however all the odd (asymmetrical modes) what effect this has had on the calculations in § 4.3 it is impossible to say.

In this example there are other factors such as the flexibility of the duct walls that are not accounted for in 'Extended'. Better agreement with the calculations was achieved upon removal of the perforated cylinder.

Comparison between calculations with 'Extended' and published literature [ref. 5] and [ref. 9] is very good. Not all the results appear in this thesis. One result in Fig. 3.10 is in excellent agreement with both Wassilieff's own calculations and measurements. Wassilieff took great care to attempt to excite single modes and preferably the $(0,0)$.

The author is therefore of the opinion that a deal of the sources

of 'error' in § 4 are not so much due to the algorithm as due to to the poor measuring conditions, see p. 55.

Obviously a lot more work needs to be done, and measurements need to be performed where one can isolate the modes.

It is suggested that improvements in the algorithm are made. There still seems to be problems with solutions appearing that in fact belong to higher order modes than that with which the iteration began, see Figs. 3.7 and 3.8. Sometimes it is easy to separate the modes see Fig. 3.10 and at other times with just a slight alteration in one of the parameters the solution drastically changes, see Figs. 3.3 (f), (g) and (h).

The author's definition of a 'jump' (see programme line 10190 p. 48) requires more investigation. A more sophisticated method may consider also discontinuities in the first differential for both $\hat{O}$ and w . In this way one may avoid situations such as shown in Fig. 3.8.

Furthermore two arrays could be calculated and the composite array, consisting of the lower of the two values for $\text{Re}\{\hat{O}\}$ at each frequency, would then at least produce the 'curve of least attenuation' although possibly consisting of two modes as in Fig. 3.10. An extra refinement might be to consider having a frequency resolution that adjusted itself to accommodate the progress or lack of it in obtaining convergence. This would avoid situations such as shown in Fig. 3.3 (g) and (h).

To finish with, the author would like to emphasise that below the cut on for the (0,2) it is always (0,0) that is the least attenuated. There exists at least one region where one can safely calculate for the propagation of sound in ducts with absorbent linings.

"Skønt er det, vi ser,
skønnere det, vi erkender
skønnest af alt er det, vi ikke fatter."

Den Salige Niels Steensen (1638-86)
Biskop og videnskabsmand

LIST OF SYMBOLS

A	normalised flow resistivity
b	ratio of lining thickness to duct width
c	the free field speed of sound 344.25 m/s (20°C)
d	the lining thickness
f	frequency
h	propagation constant in lining; ($= -j \cdot \Gamma_a$), see Γ_a
j	square root of minus one
K	bulk modulus
k	wave number
l	the duct width
M	the ratio of density of air, ρ , to the complex density, ρ' , of the lining
m	represents zero or a positive integer
n	as m above
P	atmospheric pressure, also represents porosity
p	sound pressure see also p. 11
R	the real part of characteristic impedance Z of lining
w	defined on p. 11
Z_a	characteristic impedance of lining
α	real part of propagation constant in [ref. 9] see p. 13
β	imaginary " " " " "
Γ	common propagation constant (lining and duct)
Γ_a	propagation constant of lining; ($= j \cdot h$), see h
ε	"error bound"
σ	flow resistivity
Φ	particle velocity potential (air)
θ	" " " (lining)

The meaning of other symbols is given where they occur.

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