

# **SOUND PROPAGATION IN DUCTS**

## **WITH**

# **ABSORBENT LININGS**

M.Sc. Thesis

by

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at

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Dedicated to Susanne, Martin and Maria-Therese

## Foreword

Many noise problems can be traced to sound transmitted in pipes and ducts — for example from ventilation systems, chimneys, or motor exhaust systems. Mufflers are used to limit this noise; among those widely used are absorptive mufflers, called silencers. To reduce noise from motor exhaust systems a combination of reflective mufflers and silencers is often used.

The aim of this thesis is, on the basis of a state-of-the-art study, to develop an algorithm to describe sound propagation in ducts with absorbent linings. It has been necessary to employ iterative procedures including Newton–Raphson’s method.

Calculations using the algorithm have been compared with both published calculations and measurements. The algorithm has also been tested against measurements made by the author on a silencer generously donated by Ødegaard & Danneskiold-Samsøe A/S.

Many thanks are due to my supervisor Finn Jacobsen for his active support and encouragement.

## Forord

Mange støjproblemer kan henføres til lyd transmitteret i rør eller kanaler; dette gælder f.eks. støj fra ventilations-anlæg, skorstene eller motorudstødningssystemer. For at begrænse denne støj anvendes lyddæmpere. F.eks. er det almindeligt at anvende absorptionslyddæmpere i ventilationskanaler.

Formålet med projektet er, ud fra en litteraturgennemgang, at opstille en beregningsmodel (algoritme) for lydudbredelse i et absorptionsbeklædt rør. Det viste sig at være nødvendigt at anvende iterative numeriske metoder til løsning af problemet, bl.a. Newton–Raphsons metode.

Jeg er meget taknemlig for det aktive støtte og hjælp min vejleder Finn Jacobsen har givet mig.

# 1. Introduction

For loss-free sound transmission in hard-walled ducts of uniform cross-section, the propagation constant is imaginary for modes above their cut-on frequency. When losses are included the propagation constant becomes complex, the real part representing attenuation. For modes below cut-on, the wave is evanescent. Absorption then simply adds a real number to the propagation constant.

The problem of sound propagation in ducts with absorptive linings was treated by Morse in 1939 [Ref. 2], who considered locally reacting linings. Such linings exist where viscous forces are high, or where the lining is partitioned by closely-spaced baffles that inhibit surface wave propagation.

## 1. INTRODUCTION

For loss free sound transmission in hard walled ducts of uniform cross-section the associated propagation constant is imaginary for those modes above their cut on frequency. When losses are taken into consideration the propagation constant associated with the wave becomes complex, the real part of which represents the attenuation. For those modes below their cut on frequency the associated propagation constant is always completely real and the wave is evanescent. The presence of absorption in this latter case is just to increase the attenuation by adding a real number to the propagation constant. Interestingly the transmission of sound in ducts of varying cross-section is akin to the phenomenon of barrier penetration in quantum mechanics, see [ref. 1]. Evanescent waves can penetrate narrow constrictions in ducts in a similar fashion to the penetration of particles through a potential barrier. In this latter case it is a question of an evanescent solution to Schrödinger's wave equation, within the barrier.

The models below that attempt to describe sound absorption have as their aim the determination of the complex propagation constant.

The problem of sound propagation in ducts with absorptive linings was dealt with by Morse in 1939 [ref. 2] where he considered the problem from the point of view of locally reacting linings, this is where the effect of the lining on the passage of sound can be wholly described in terms of its acoustic impedance, Scott, [ref. 3; § 6]. Such linings exist where the viscous forces are high or where the impedance is much greater than that of air. The angle of incidence from air into the lining is very nearly zero. Local reaction also occurs where the lining is separated by closely spaced partitions or baffles that inhibit wave propagation along the duct lining, see [ref. 1] and Fig. 1.1 below.

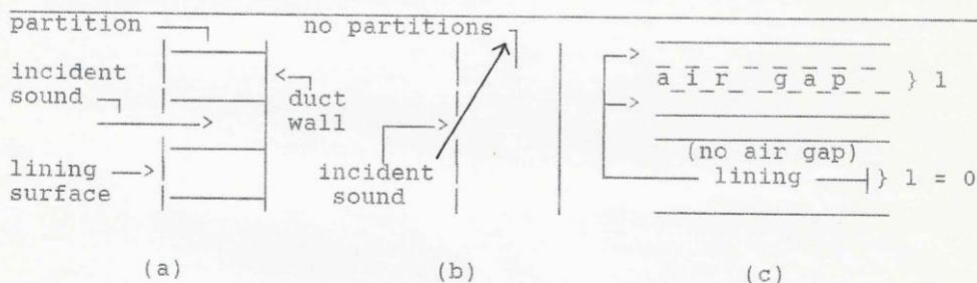


Fig. 1.1. (a) Locally reacting linings. (b) Non locally reacting linings. (c) The assumption that the duct liner in (b) behaves like the liner in (a) leads to the incorrect conclusion that the attenuation becomes infinite as the width of the air gap,  $l$ , goes to zero.

A simple demonstration as indicated in Fig. 1.1 (c) shows that the assumption of local reaction for liners leads to the incorrect

Figure 1.1. (a) Locally reacting linings. (b) Non-locally reacting linings. (c) Assuming local reaction leads to the incorrect conclusion that attenuation  $\rightarrow \infty$  as the air-gap width  $l \rightarrow 0$ .

Scott [Ref. 3] (1946) was the first to develop a theory for extended (or bulk) reaction, in which wave propagation in both lining and air space shares a common propagation constant. Christie [Ref. 5] (1971) solved both the Morse and Scott equations for a range of flow resistivities using a Newton–Raphson method.

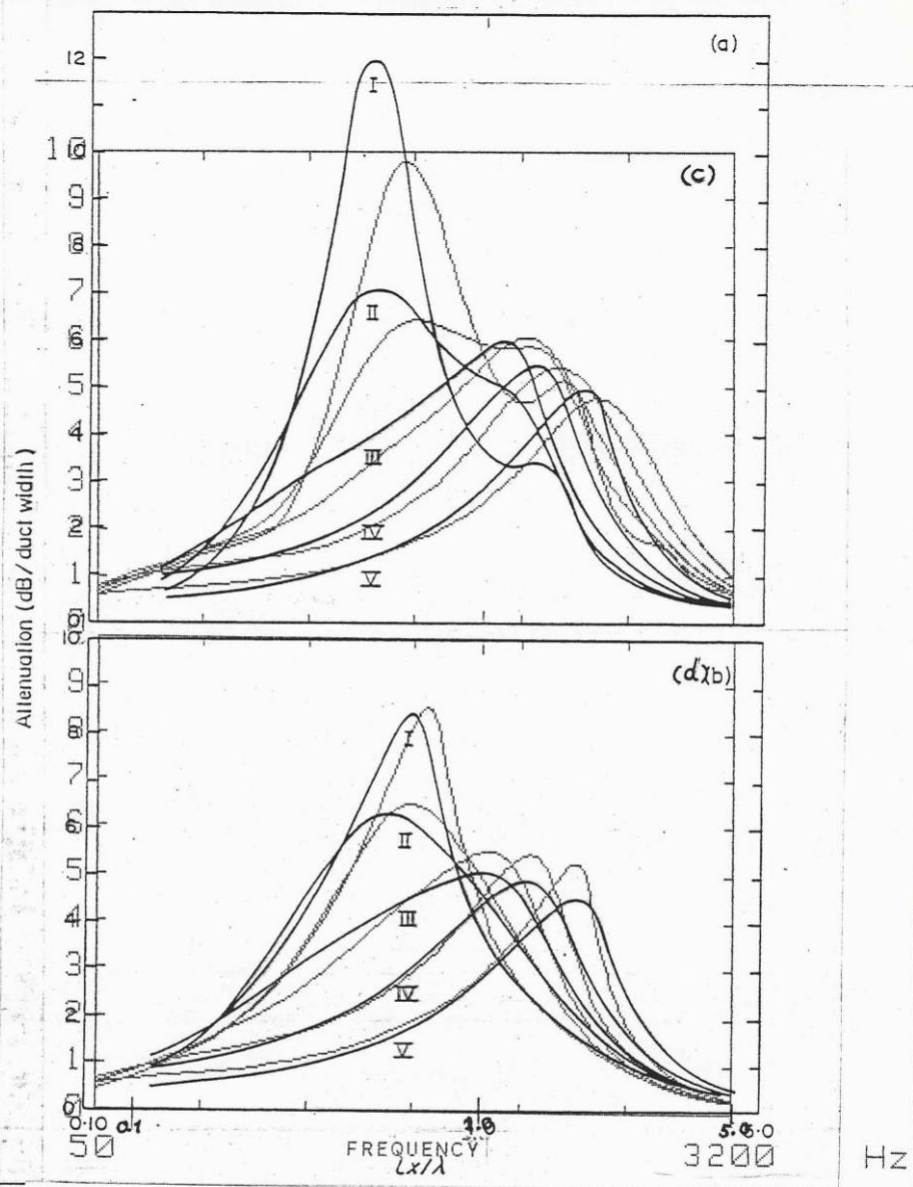


Fig. 1.2. The transparency is taken from Christie [ref. 5].  
 (a) Morse's analysis see eq. 2.2.17  
 (b) Scott's analysis see eq. 2.2.9  
 (c) Morse's analysis using the 'Four\_pole' programme  
 (corrected for rectangular ducts using eq. 2.1.3)  
 (d) Scott's analysis using the algorithm 'Extended'

Duct width,  $l = 0.5334$  m; lining thickness  $d = 0.1524$  m

|                  | I    | II   | III   | IV    | V     |
|------------------|------|------|-------|-------|-------|
| Flow Resistivity | 4000 | 8000 | 16000 | 32000 | 64000 |

Figure 1.2. (a) Morse's analysis (eq. 2.2.17). (b) Scott's analysis (eq. 2.2.9). (c) Morse using the 'Four\_pole' programme. (d) Scott using 'Extended'. Duct width  $l = 0.5334$  m,  $d = 0.1524$  m. Flow resistivities I-V: 4000, 8000, 16000, 32000, 64000 N·s/m⁴.

Building on Christie's work, this thesis presents the algorithm 'Extended' for solving extended reaction, used with the four-pole programme devised by Finn Jacobsen [Refs. 7 & 8]. Scott's theory is outlined in Section 2; the algorithm in Section 3; measurements in Section 4; conclusions in Section 5.

## 2. Theory

### 2.1 General Considerations

The theory throughout this thesis is that of Scott [Ref. 3; § 3], assuming a rectangular duct. By symmetry the component of particle velocity normal to the central plane is zero, limiting the analysis to modes (0, n) where n = 0, 2, 4, ...

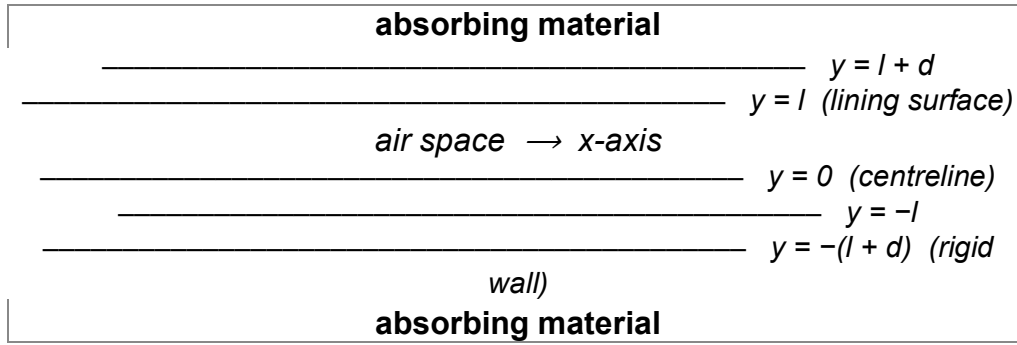


Figure 2.1. Plan of channel [Ref. 3]. Air space:  $-l \leq y \leq l$  (half-width  $l$ ). Absorbing material:  $l \leq y \leq l+d$  (thickness  $d$ ), and symmetrically below. Sound propagates in the  $x$ -direction.

The porous material properties are described by the Delany–Bazley formulae [Ref. 6] modified by Mechel [Ref. 12], with normalised flow resistivity  $A = \sigma/(\rho f)$ :

**For  $A \leq 60$ :**

$$\Gamma a / k = 0.189 \cdot A^{0.6185} + j[1 + 0.0978 \cdot A^{0.6929}] \quad (2.1.1a)$$

$$Za/Z_0 = 1 + 0.04885 \cdot A^{0.754} - j \cdot 0.087 \cdot A^{0.7307} \quad (2.1.1b)$$

**For  $A \geq 60$ :**

$$\Gamma a / k = [-1.4664 + j \cdot 0.2115 \cdot A]^{1/2} \quad (2.1.2a)$$

$$Za/Z_0 = \left( \frac{A}{2\pi} + j \cdot 1.4035 \right) / (\Gamma a / k) \quad (2.1.2b)$$

Here  $\Gamma a$  is the lining propagation constant,  $Za$  is the lining characteristic impedance,  $Z_0 = \rho c$ ,  $k = \omega/c$ ,  $\sigma$  is flow resistivity,  $\rho$  is the density of air, and  $f$  is frequency.

For a square duct ( $W = H$ ) with  $2R = H$ , the real part of the propagation constant satisfies:

$$\text{Re}\{\Gamma(p)\} = 2 \cdot \text{Re}\{\Gamma(s)\} = 2 \cdot \text{Re}\{\Gamma(c)\} \quad (2.1.3)$$

(This factor of 2 is accounted for in programme lines 10960–11060 of ‘Extended’.)

#### 2.1.1 Some Comments on the Four-Pole Method

The four-pole matrix [Ref. 7; § 4] is mathematically identical to the Lorentz (Mach) transformation, which leaves the wave equation invariant. The Mach transformation can also be applied to duct acoustics when flow is present.

### 2.2 Scott's Theory

The field equations for the air space and lining respectively are:

$$v = -\nabla\Phi, p = j\omega\rho\Phi, \nabla^2\Phi + k^2\Phi = 0 \quad (2.2.1a)$$

$$v = -\nabla\Theta, p = j\omega\rho'\Theta, \nabla^2\Theta + h^2\Theta = 0 \quad (2.2.1b)$$

Scott proposed that for any one mode,  $\Phi$  and  $\Theta$  share a common propagation constant  $\Gamma$ , giving for a rectangular duct:

$$\Phi = \Phi(y) \cdot e^{-\Gamma x} \cdot e^{j\omega t} \quad (2.2.2a)$$

$$\Theta = \Theta(y) \cdot e^{-\Gamma x} \cdot e^{j\omega t} \quad (2.2.2b)$$

Substituting into the field equations:

$$\Phi(y) = A \cdot e^{j\lambda y} + B \cdot e^{-j\lambda y}; \lambda^2 = \Gamma^2 + k^2 \quad (2.2.3a)$$

$$\Theta(y) = C \cdot e^{j\mu y} + D \cdot e^{-j\mu y}; \mu^2 = \Gamma^2 + h^2 \quad (2.2.3b)$$

The symmetry condition at  $y = 0$  gives  $A = B$ . The rigid outer wall at  $y = l+d$  gives:

$$D = C \cdot e^{2j\mu(l+d)} \quad (2.2.4)$$

Continuity of normal particle velocity and pressure at the lining–air surface  $y = l$  yields:

$$\rho \cdot \frac{\coth(-j\lambda l)}{\lambda} = \rho' \cdot \frac{\coth(j\mu d)}{\mu} \quad (2.2.6)$$

Using Equation 2.2.3 this becomes:

$$\rho \cdot (\Gamma^2 + h^2)^{1/2} \cdot \tan\{(\Gamma^2 + h^2)^{1/2} \cdot d\} = -\rho' \cdot (\Gamma^2 + k^2)^{1/2} \cdot \tan\{(\Gamma^2 + k^2)^{1/2} \cdot l\} \quad (2.2.7)$$

Christie [Ref. 5] introduces the variables:

$$w = ((\Gamma l)^2 + (kl)^2)^{1/2}, p = ((hl)^2 - (kl)^2)^{1/2}, b = d/l, M = \rho/\rho' \quad (2.2.8)$$

allowing Equation 2.2.7 to be rewritten as:

$$f(w) = w \cdot \tan w + M \cdot [w^2 + p^2]^{1/2} \cdot \tan(b \cdot \sqrt{w^2 + p^2}) = 0 \quad (2.2.9)$$

This transcendental equation must be solved iteratively.

### 2.2.1 Properties of the Lining

The boundary condition of equal normal particle velocity at the lining–air interface gives:

$$\partial p / \partial y = (\rho / \rho') \cdot \partial p' / \partial y \quad (2.2.10)$$

Scott [Ref. 4; § 2] shows the complex density is:

$$\rho' = h^2 \kappa / (P \omega^2) \quad (2.2.11)$$

where  $\kappa$  is the effective compressibility of pore air and  $P$  is porosity. The characteristic impedance is:

$$Za = \rho' \omega / h \quad (2.2.12)$$

These give:

$$\partial p / \partial y = (\rho \kappa / Za \cdot h) \cdot \partial p' / \partial y \quad (2.2.13)$$

Writing  $\Gamma a = jh = \alpha + j\beta$ , the ratio  $M = \rho/\rho'$  in Christie's equation becomes:

$$M = -j\rho \kappa / (Za \Gamma a)$$

This corresponds to programme line 9230 in 'Extended'.

### **2.2.2 Low and High Frequency Values of $w$**

From Equations 2.1.1 and 2.1.2 with  $A = \sigma/(\rho f)$  and  $k = 2\pi f/c$ , one finds that as  $f \rightarrow 0$ ,  $\Gamma a \rightarrow 0$  but  $M \rightarrow 0$ , so Scott's equation reduces to Morse's equation:

$$w \cdot \tan w = -jkl / Za \quad (2.2.17)$$

Solutions to the (0,0) mode start at  $w = 0 + j \cdot 0$  and solutions to the (0,2) mode start at  $w = \pi + j \cdot 0$ . The high-frequency end-point for the least-attenuated mode is  $w = \frac{1}{2}\pi + j \cdot 0$ .

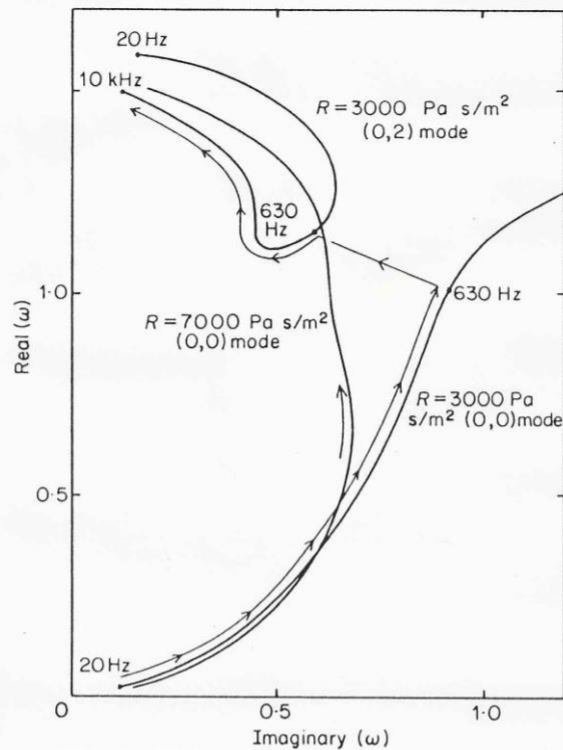


Fig. 2.2. Reproduced from [ref. 9]. Plot of  $\text{Re}\{w\}$  versus  $\text{Im}\{w\}$  calculated from eq. (7), [ref. 9] (in the isotropic case here, this is identical with eq. 2.2.9). Duct width  $l = 0.3$  m lining thickness  $d = 0.15$  m, flow resistivities ( $R$  in fig. 3)  $\sigma = 3000$  N·s/m<sup>4</sup> and 7000 N·s/m<sup>4</sup>. According to Wassilieff the arrows follow the roots yielding least attenuation.

The author has found that the high frequency end point for the least attenuated mode (either (0,0) or (0,2)) is  $\frac{1}{2}\pi + j \cdot 0$ , this is in agreement with Wassilieff [ref. 9] and plate IV in [ref. 10]. Furthermore, calculations with 'Extended' show that the (0,2) mode can be the least attenuated mode. This is borne out by Wassilieff, in Fig. 3 (Fig. 2.2 in this thesis) and Fig. 4a. of [ref. 9].

Concerning low frequency values of  $w$ , Wassilieff's argument holds if  $\Gamma_a$  and  $k$  approach zero at the same rate and in such a way that their ratio approaches  $0 + j$ , see eq. 2.1.1. It is shown below that this is not the case and that in the low frequency limit  $M$  goes to zero (i.e.  $\varphi'$  does not approach  $\varphi$ ) and that  $\pi + j \cdot 0$  is the starting point for  $w$  for the (0,2) mode as is also the case with Morse's equation for locally reacting linings.

An examination of Delany and Bazley's formulae as given in § 2,

Figure 2.2. Reproduced from [Ref. 9].  $\text{Re}\{w\}$  versus  $\text{Im}\{w\}$  from eq. 2.2.9. Duct width  $l = 0.3$  m,  $d = 0.15$  m. Flow resistivities  $\sigma = 3000$  and  $7000$  N·s/m<sup>4</sup>.

The following simple numerical example taken from Fig. 2.2 (Fig. 3 in [ref. 9]) serves to highlight the above.

From Fig. 2.2 where  $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}^4$  Wassilieff has, for the (0,2) mode,  $w$  starting (when  $f = 20 \text{ Hz}$ ) near to  $\frac{1}{2}\pi + j\cdot 0$  for the reasons given at the beginning of § 2.2.2. However these values for  $\sigma$  and  $f$  give with the density of air  $\rho = 1.205 \text{ kg}/\text{m}^3$  a value  $A = 124.451$ . One has therefore to use eq. 2.1.2 ( $A \geq 60$ ), assuming validity at this frequency. With  $c = 344.246 \text{ m}/\text{s}$  one obtains the following:

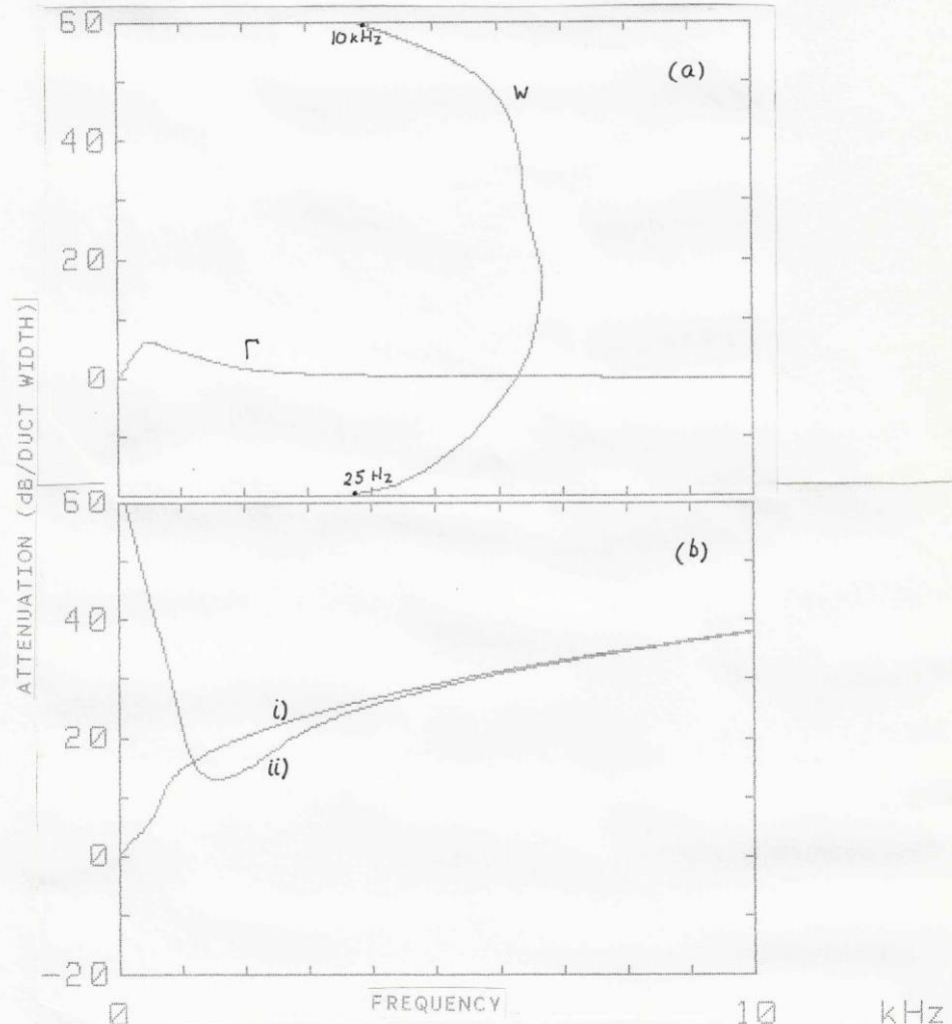


Fig. 2.3. Duct width  $l = 0.3 \text{ m}$ ; lining thickness  $d = 0.15 \text{ m}$   
 (a)  $\sigma = 7000 \text{ N}\cdot\text{s}/\text{m}^4$  and  $\Gamma$  is the attenuation curve (0,0)  
 $w$  is the Nyquist plot.  
 (b)  $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}^4$ ; i) is the (0,0) mode and  
 ii) is the (0,2) mode 'calculation'.

Figure 2.3. Duct width  $l = 0.3 \text{ m}$ ;  $d = 0.15 \text{ m}$ . (a)  $\sigma = 7000 \text{ N}\cdot\text{s}/\text{m}^4$ :  $\Gamma$  is the (0,0) attenuation;  $w$  is the Nyquist plot. (b)  $\sigma = 3000 \text{ N}\cdot\text{s}/\text{m}^4$ : curve (i) is (0,0), curve (ii) is (0,2).

$$k = 0.365; \quad \Gamma_a = 1.288 + j \cdot 1.362;$$

$$Z_a/\rho c = 2.849 - j \cdot 2.615; \quad M = 0.004 + j \cdot 0.050.$$

As can be seen  $M \approx 0$  and Scott's equation can be replaced by Morse's and as with Morse's equation the (0,2) mode begins at  $\pi + j \cdot 0$  (as is borne out by the author's own work with 'Extended', see p. 12) and not  $\frac{1}{2}\pi + j \cdot 0$  as shown in Fig. 2.2.

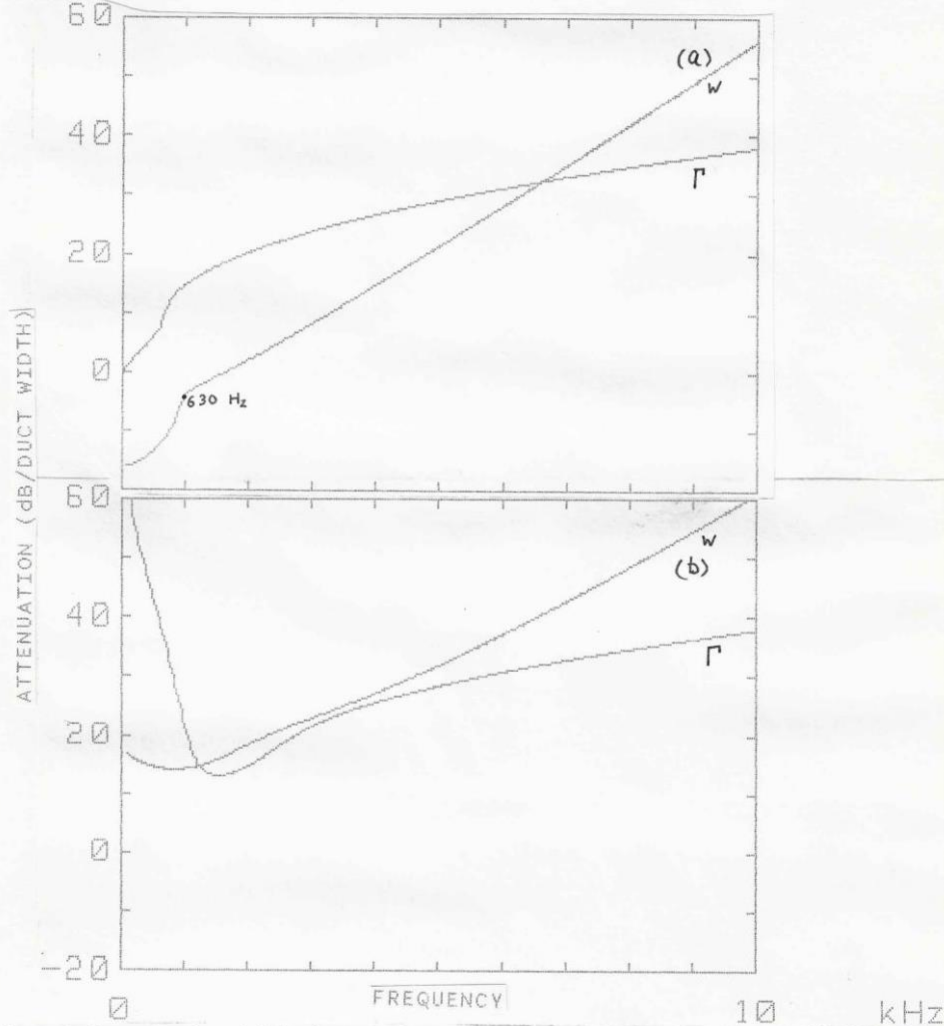


Fig. 2.4. Duct width  $l = 0.3$  m; lining thickness  $d = 0.15$  m  
 (a)  $\sigma = 3000$  N·s/m<sup>4</sup> and  $\Gamma$  is the attenuation curve (0,0)  
 $w$  is the Nyquist plot.  
 (b)  $\sigma = 3000$  N·s/m<sup>4</sup> and  $\Gamma$  is the attenuation curve (0,2)  
 $w$  is the Nyquist plot.

Figure 2.4. Duct width  $l = 0.3$  m;  $d = 0.15$  m;  $\sigma = 3000$  N·s/m<sup>4</sup>. (a) (0,0) mode; (b) (0,2) mode.

## 3. The Algorithm 'Extended'

### 3.1 Introduction

The purpose of the algorithm is to solve the transcendental Equation 2.2.9, finding values of  $w$  for which  $f(w) = 0$ . Once found, the values of  $\Gamma$  are extracted using the relations in Equation 2.2.8 (programme lines 10750 and 10840).

The solutions to  $f(w) = 0$  lie at the bottoms of valleys in the three-dimensional modulus surface  $|f(w)|$  plotted over the complex  $w$ -plane. Each mode of propagation has its own valley.

Starting from solutions to Morse's equation as initial approximations, the process is continued using the solution at one frequency as the initial value for the next step. In 'Extended' this was replaced by linear extrapolation of  $\Gamma(J)$  to obtain  $w'(J+1)$ , which proved more reliable.

A 'jump' is identified when the most recent step in  $w$  is more than three times longer than the previous step (programme line 10190). When this occurs the alternative iterative method is tried. If both methods fail, the algorithm linearly extrapolates  $w$  itself rather than  $\Gamma$ .

Error bounds used:  $\varepsilon = 10^{-15}$  for Newton–Raphson;  $\varepsilon = 10^{-10}$  for Muller's method.

### 3.2 The Newton–Raphson Method

The iteration is  $w_{n+1} = w_n - f(w_n)/f'(w_n)$ , where:

$$f(w) = w \cdot \tan w + M \cdot [w^2 + p^2]^{1/2} \cdot \tan(b \cdot \sqrt{w^2 + p^2}) = 0 \quad (3.2.1)$$

$$f'(w) = \frac{2w + \sin 2w}{1 + \cos 2w} + M \cdot \frac{w}{\sqrt{w^2 + p^2}} \cdot \frac{2b \cdot \sqrt{w^2 + p^2} + \sin(2b \cdot \sqrt{w^2 + p^2})}{1 + \cos(2b \cdot \sqrt{w^2 + p^2})} \quad (3.2.2)$$

### 3.3 Muller's Method

Writing  $f(w) = w \cdot \tan w$  and  $g(w) = -M[w^2 + p^2]^{1/2} \cdot \tan(b[w^2 + p^2]^{1/2})$ , the iteration used is:

$$f^{-1}(g(w_n)) = w_{n+1} = (g(w_n)) = \arctan\left(\frac{g(w_n)}{w_{n+1}}\right) \quad (3.3.1)$$

using the complex arctangent:

$$\arctan z = -\frac{j}{2} \cdot \ln\left(\frac{j+z}{j-z}\right) \quad (3.3.2)$$

Writing  $z = Y_1 + jY_2$ , with  $j+z = J_1 e^{jJ_2}$  and  $j-z = J_3 e^{jJ_4}$ :

$$J_1 = (Y_1^2 + (Y_2 + 1)^2)^{1/2}, J_2 = \arctan\left(\frac{Y_1}{Y_2 + 1}\right) \quad (3.3.3)$$

$$J_3 = (Y_1^2 + (1 - Y_2)^2)^{1/2}, J_4 = \arctan\left(\frac{-Y_1}{1 - Y_2}\right) \quad (3.3.4)$$

Selecting the branch via integer  $k$  (programme line 8980):

$$\arctan z = k\pi + \frac{J_4 - J_2}{2} + j \cdot \frac{\ln(J_1/J_3)}{2}$$

### 3.4 Calculations

Figure 3.1 repeats Figure 1.2(d) at different frequency resolutions; Figure 3.2 extends the flow-resistivity range and shows 'anomalous behaviour' for  $\sigma$  outside 3000–128000 N·s/m<sup>4</sup> where the (0,0) mode attenuation rises without limit.

that for these duct dimensions ( $l = 0.5334$  m and  $d = 0.1524$  m) the (0,0) mode seems to show 'anomalous behaviour', outside the range  $3,000 \text{ N}\cdot\text{s/m} < \sigma < 128,000 \text{ N}\cdot\text{s/m}$ , in that the attenuation continues to rise without limit with increasing frequency.

The series of figures in Fig. 3.3 show a parameter study of curve no. 9 in Fig. 3.2. It is demonstrated for  $\sigma = 32,000 \text{ N}\cdot\text{s/m}$  (for the duct dimensions above) that the (0,2) mode is the least attenuated mode above frequencies for which this mode is cut on provided the lining thickness is less than 4 cm.

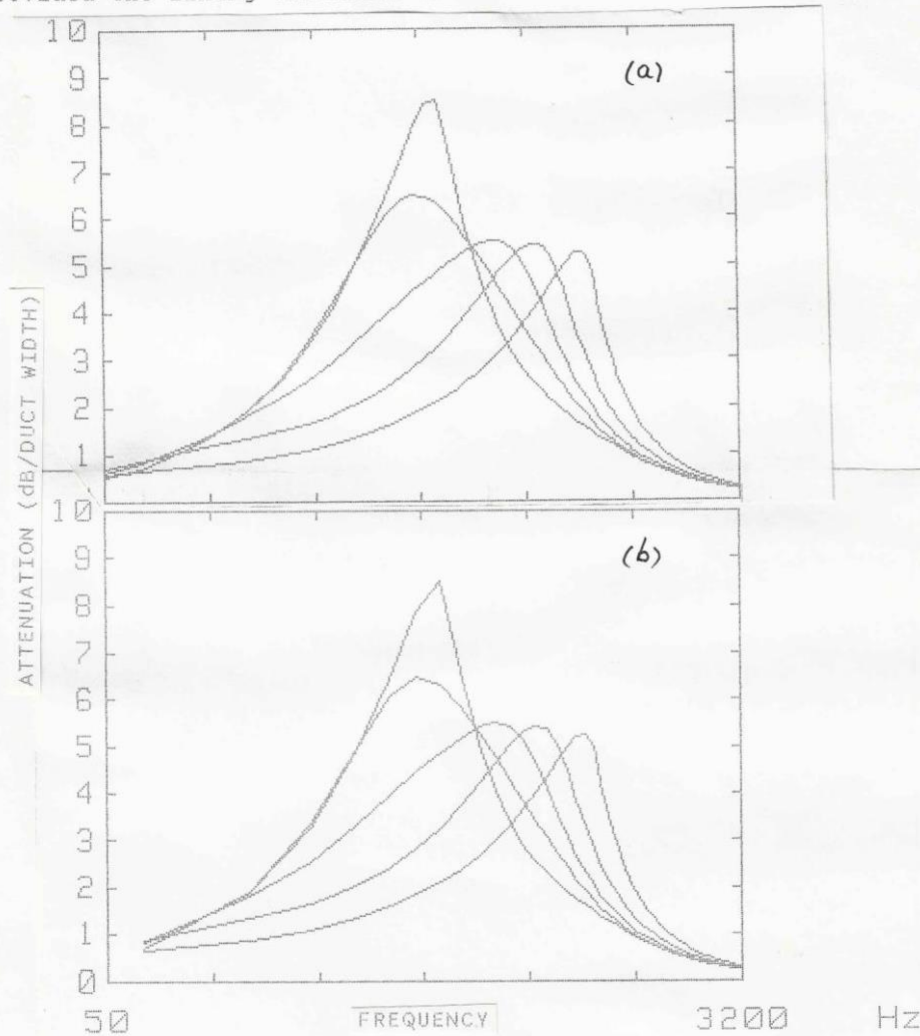


Fig. 3.1. To be compared with Fig. 1.2 (d) where the frequency resolution ( $F_{res}$ ) is 16 Hz.  
(a)  $F_{res} = 32$  Hz; (b)  $F_{res} = 64$  Hz.

Figure 3.1. Comparison of frequency resolutions. Duct width  $l = 0.5334$  m;  $d = 0.1524$  m. (a)  $F_{res} = 32$  Hz; (b)  $F_{res} = 64$  Hz.

Fig. 3.3 (f) is examined in closer detail in Fig. 3.4. It is shown that the two modes  $(0,0)$  and  $(0,2)$  do cross each other and that for increasing frequency the  $(0,2)$  mode is the least attenuated.

The influence of choice of frequency resolution is dramatically demonstrated in Fig. 3.3 (g) and Fig. 3.3 (h). In the latter the proper behaviour of the  $(0,2)$  mode is brought out. Fig. 3.5 is a closer inspection of Fig. 3.3 (h). The difficulties encountered around the cut on frequency for  $(0,2)$  are clear to see.

The behaviour of modes  $(0,0)$  and  $(0,2)$  for curve nos. 4, 5 and 11 in Fig. 3.2 is examined in Figs. 3.6 and 3.7 and Fig. 3.8.

This analysis is continued in Fig. 3.9 for curve no. 12 in Fig. 3.2. It is demonstrated that again in this case the  $(0,2)$  mode is the least attenuated mode above cut on.

This behaviour was unexpected by the author who at first refused to believe these results assuming that the 'error' lay with the algorithm. This behaviour, where the  $(0,0)$  mode is not always the least attenuated and which the author describes as anomalous, should be considered in the light of the remarks on p. 5 of this thesis and Wassilieff's comments in [ref. 9].

The four-pole approach to duct acoustics is based on the assumption of plane wave propagation. The consequences to this approach in view of the above has not yet been fully considered.

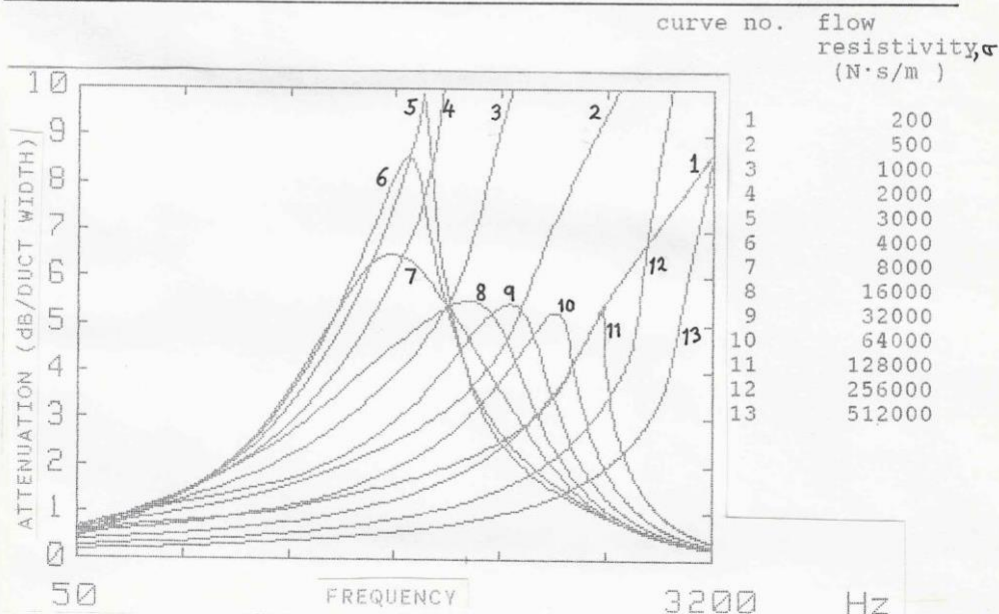


Fig. 3.2. Duct width  $l = 0.5334$  m and lining thickness  $d = 0.1524$  as in [ref. 5] and Fig. 1.2. The flow resistivity for each curve is as indicated above.

Figure 3.2. Duct width  $l = 0.5334$  m;  $d = 0.1524$  m. Attenuation curves for flow resistivities from 200 to 512000 N·s/m<sup>4</sup> (curves 1–13).

Figure 3.3 shows a parameter study for  $\sigma = 32000 \text{ N}\cdot\text{s}/\text{m}^4$  with varying lining thickness, demonstrating that the (0,2) mode is the least attenuated above its cut-on frequency when lining thickness is less than 4 cm.

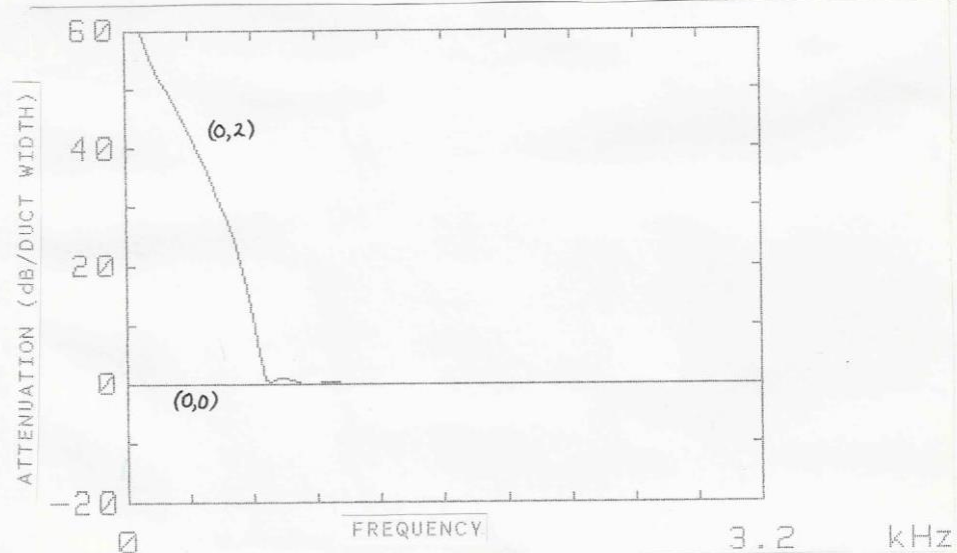


Fig. 3.3. (a) Duct width  $l = 0.5334 \text{ m}$ ; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000 \text{ N}\cdot\text{s}/\text{m}^4$ .

Lining thickness  $d = 0.0001 \text{ m}$ .

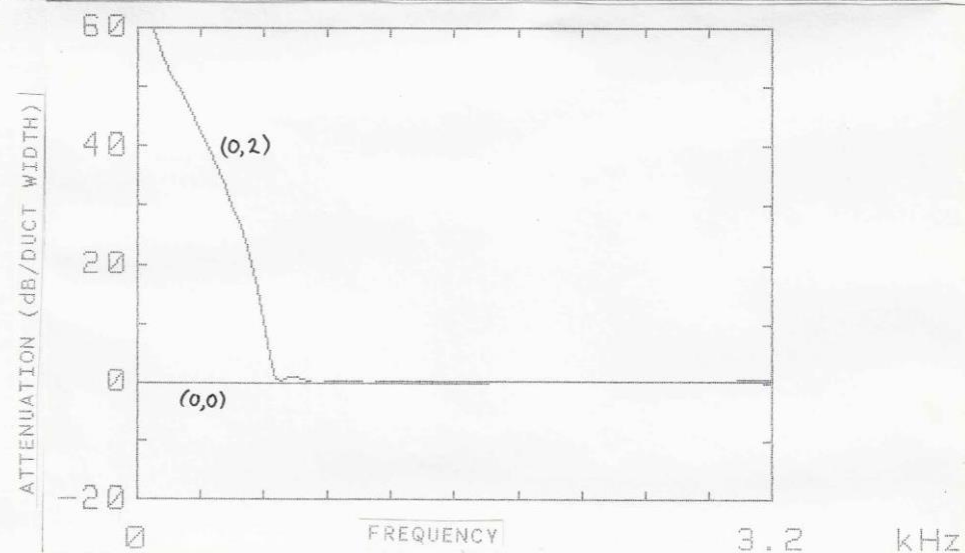


Fig. 3.3. (b) Duct width  $l = 0.5334 \text{ m}$ ; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000 \text{ N}\cdot\text{s}/\text{m}^4$ .

Lining thickness  $d = 0.0010 \text{ m}$ .

Figure 3.3(a).  $d = 0.0001$  m.  $\sigma = 32000$  N·s/m<sup>4</sup>;  $l = 0.5334$  m.

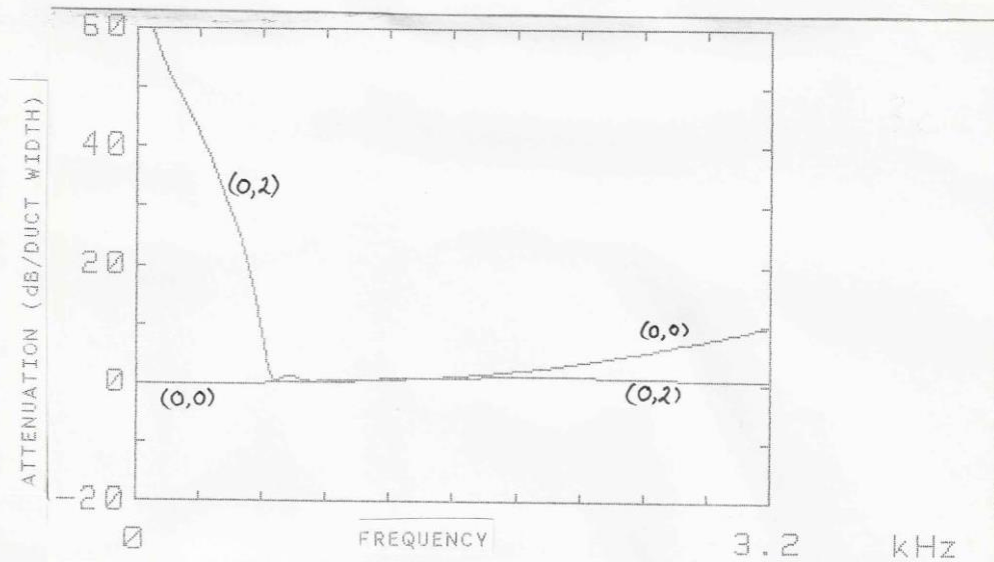


Fig. 3.3. (c) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .

Lining thickness  $d = 0.0050$  m.

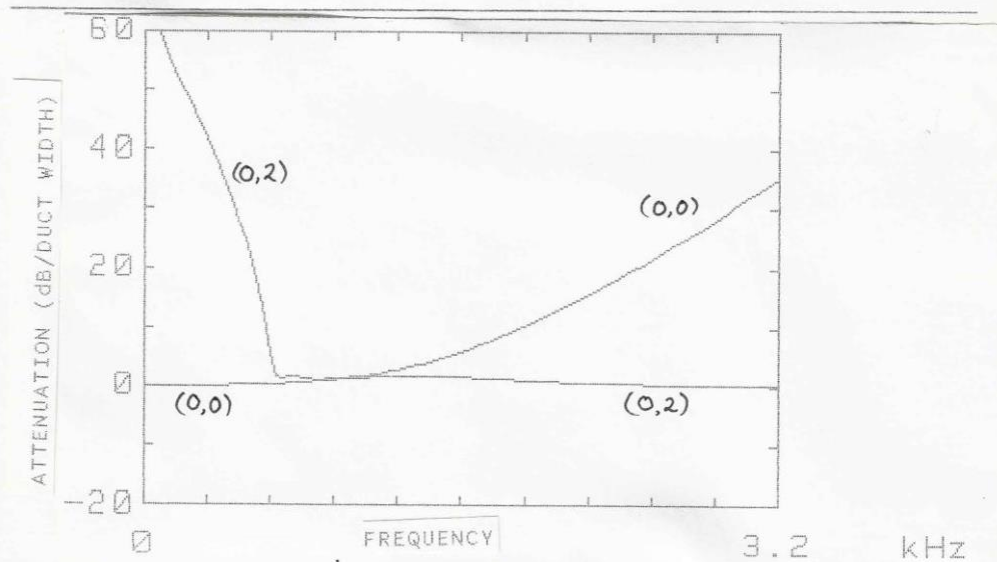


Fig. 3.3. (d) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .

Lining thickness  $d = 0.0100$  m.

Figure 3.3(b)–(d).  $d = 0.001, 0.005, 0.010$  m.

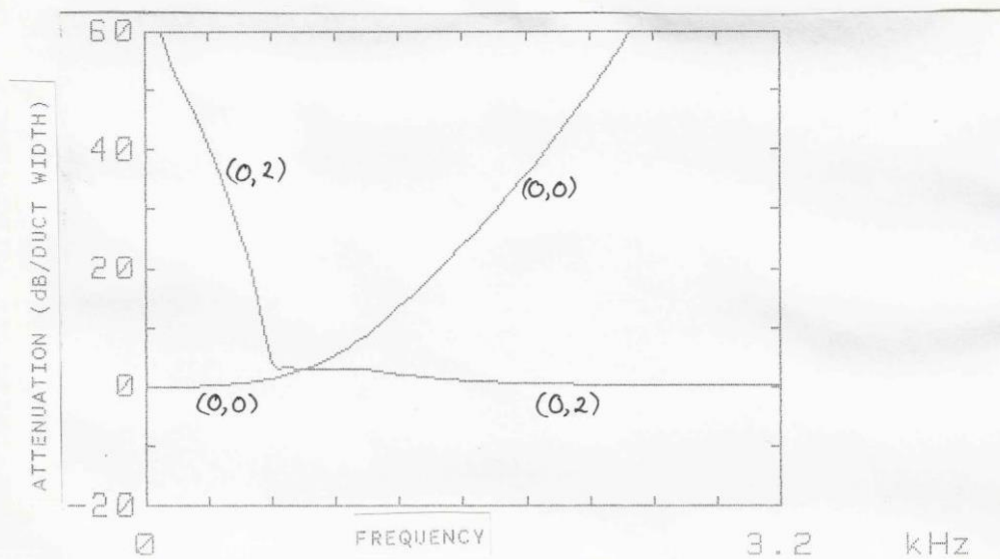


Fig. 3.3. (e) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .

Lining thickness  $d = 0.0200$  m.

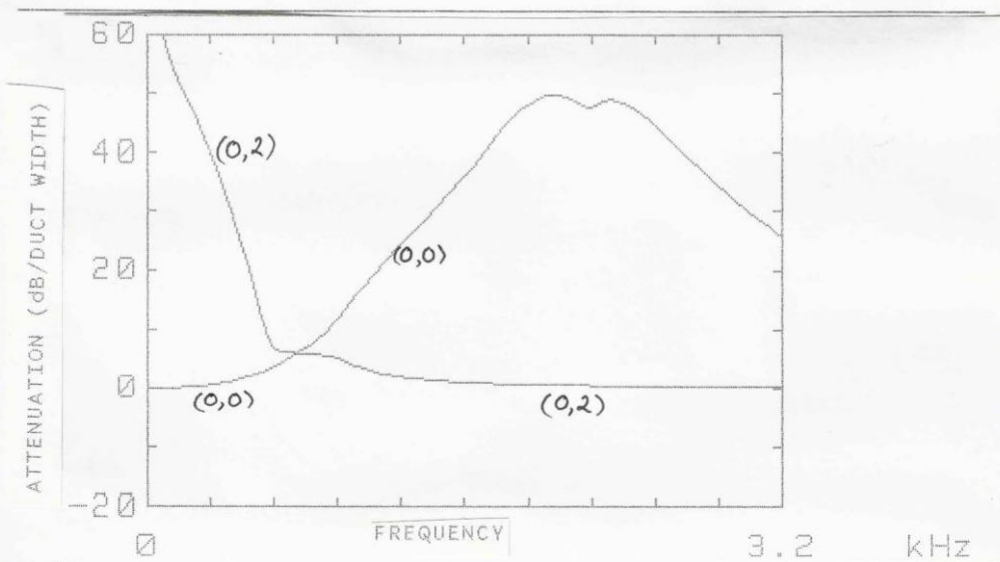


Fig. 3.3. (f) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .

Lining thickness  $d = 0.0300$  m.

Figure 3.3(e)-(f).  $d = 0.020, 0.030$  m.

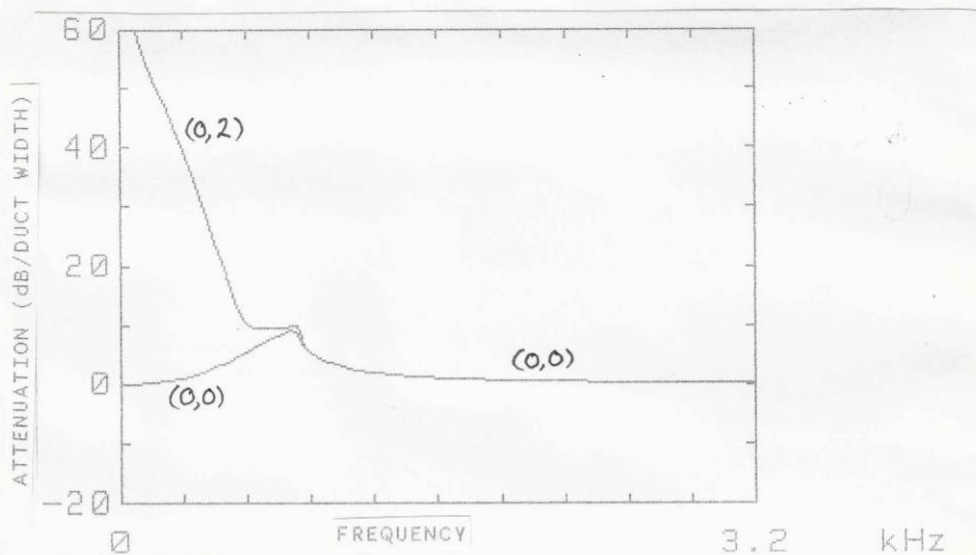


Fig. 3.3. (g) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .  
Lining thickness  $d = 0.0400$  m.

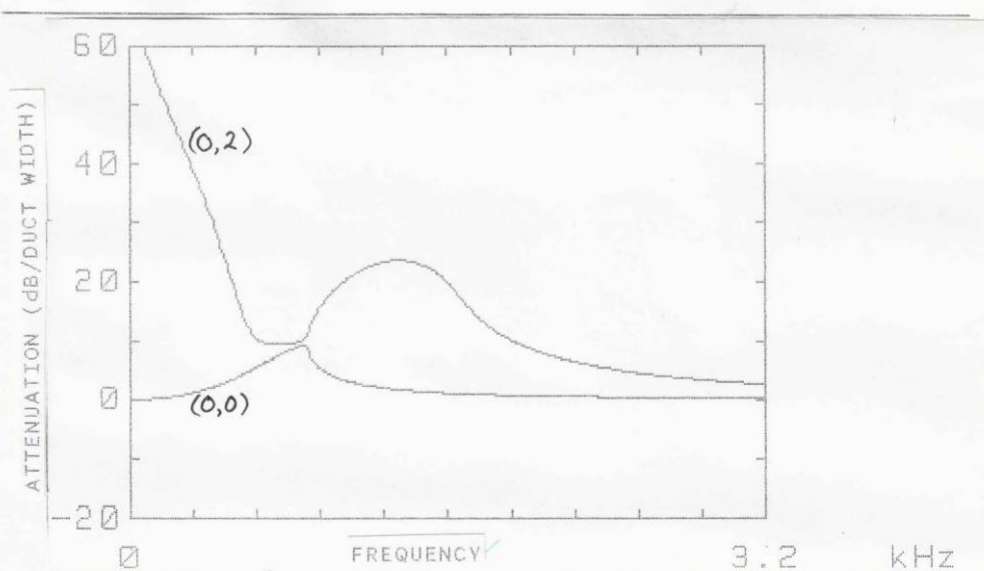


Fig. 3.3. (h) Duct width  $l = 0.5334$  m;  
Flow resistivity  $\sigma = 32,000$  N·s/m .  
Lining thickness  $d = 0.0400$  m. Freq. res. = 8.06 Hz

Figure 3.3(g)–(h).  $d = 0.040$  m (at 32 Hz and 8.06 Hz frequency resolution).

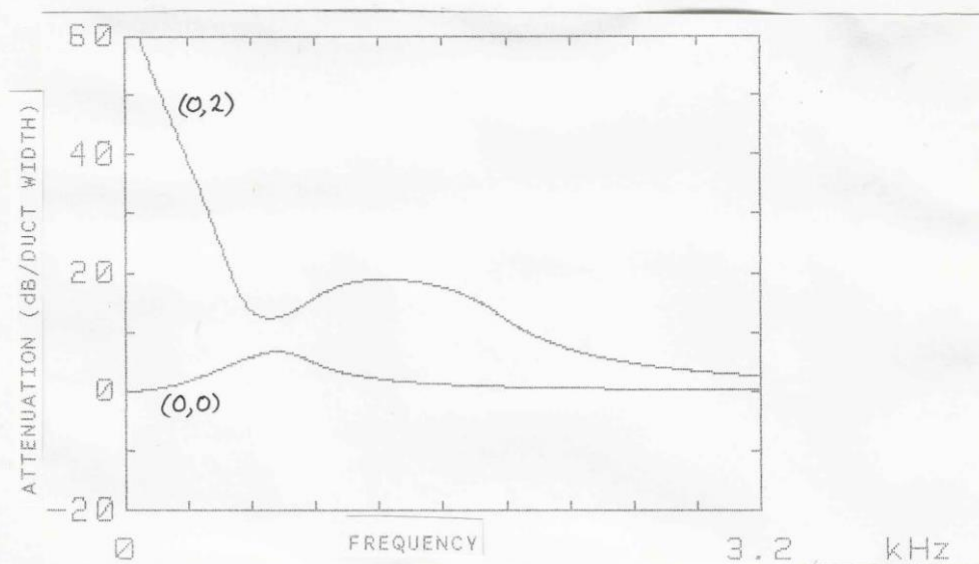


Fig. 3.3. (i) Duct width  $l = 0.5334$  m;  
Flow resistivity  $\sigma = 32,000$  N·s/m .  
Lining thickness  $d = 0.0500$  m. Freq. res. = 8.06 Hz

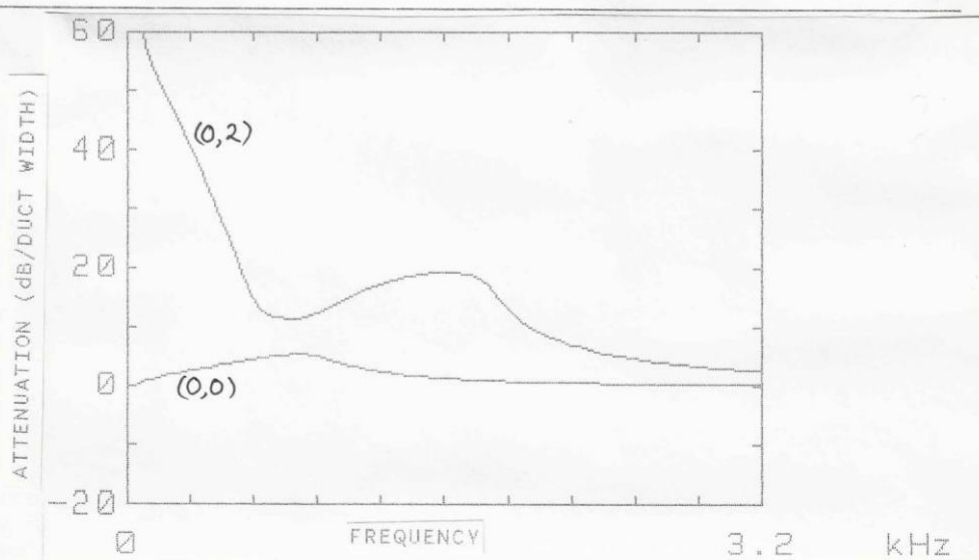


Fig. 3.3. (j) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .  
Lining thickness  $d = 0.1000$  m.

Figure 3.3(i)-(j).  $d = 0.050, 0.100$  m.

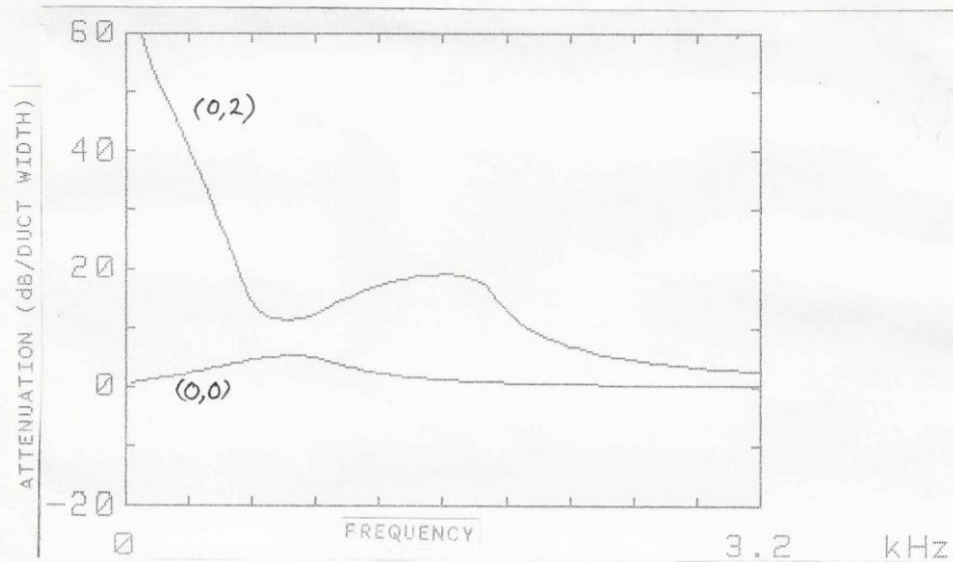


Fig. 3.3. (k) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .  
Lining thickness  $d = 0.2000$  m.

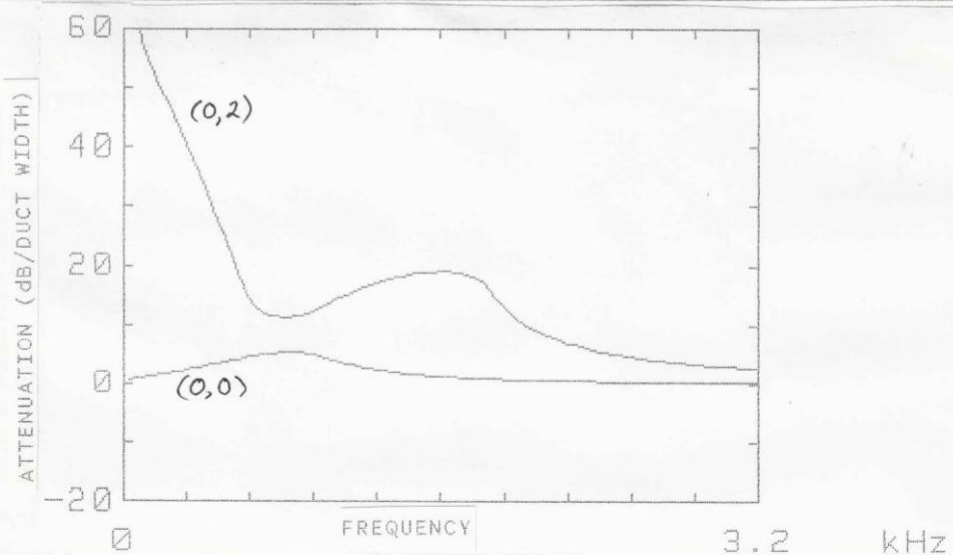


Fig. 3.3. (l) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m .  
Lining thickness  $d = 0.3000$  m.

Figure 3.3(k)-(l).  $d = 0.200, 0.300$  m.

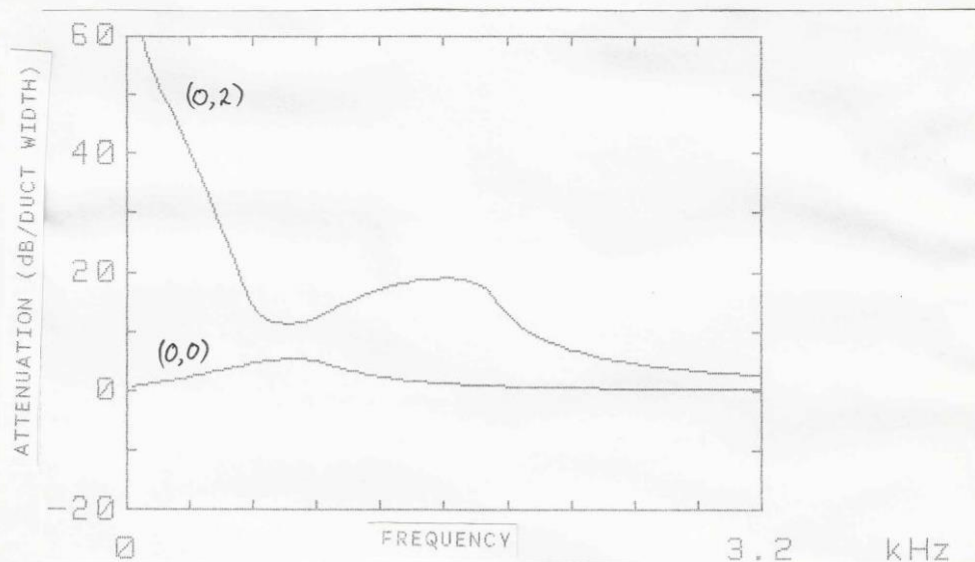


Fig. 3.3. (m) Duct width  $l = 0.5334$  m; Freq. resolution = 32 Hz  
Flow resistivity  $\sigma = 32,000$  N·s/m.

Lining thickness  $d = 1.0000$  m.

The attenuation curves in Figs. 3.3 (f), (g) and (h) are now examined in greater detail. Fig. 3.4 shows the curves for (0,0) and for (0,2) with their Nyquist diagrams. A print out of the arrays for  $w$  showed the following:

Fig. 3.4 (a) (0,0) mode;

$f = 32$  Hz;  $\text{Re}\{w\} = 0.0008$ ;  $\text{Im}\{w\} = 0.0602$

$f = 3.2$  kHz;  $\text{Re}\{w\} = 10.6493$ ;  $\text{Im}\{w\} = 1.6016$

Fig. 3.4 (b) (0,2) mode;

$f = 32$  Hz;  $\text{Re}\{w\} = 3.1403$ ;  $\text{Im}\{w\} = -0.0027$

$f = 3.2$  kHz;  $\text{Re}\{w\} = 1.5658$ ;  $\text{Im}\{w\} = 0.1376$

Negative values for  $\text{Im}\{w\}$  as in the example above, occur very seldom and then only very close to zero when using Muller's method. The author believes that this is an example of the 'solution' oscillating about the correct value. In that case the real accuracy (not  $\text{En}$ ) can, in this case, be seen to be approximately  $\pm 0.003$ . Only when  $f > 512$  Hz does  $\text{Im}\{w\}$  begin to exceed 0.09 in curve (b).

Further examination of the print out revealed a very rapid decline

Figure 3.3(m).  $d = 1.000$  m.

in  $\text{Im}\{w\}$  (curve (a)) for frequencies just below 3.2 kHz.  $\text{Im}\{w\}$  is not approaching  $3\frac{1}{2}\pi + j\cdot\frac{1}{2}\pi$  but is in fact approaching  $3\frac{1}{2}\pi + j\cdot 0$ . According to [ref. 9]  $\text{Im}\{w\} \rightarrow 0$  in the limit of increasing frequency and that the high frequency end point for  $w$  is always given by [ref. 9]:

$$w = \frac{1}{2} \cdot m\pi + j \cdot 0; m = 0, 1, 3, 5 \dots$$

3.3.5

The author has never found a high frequency end point with  $m = 0$ , and further the high frequency end point for  $w$  is not always given as shown above, see for example curve  $w$  in Fig. 2.4 (a) and compare with Fig. 2.2 from [ref. 9].

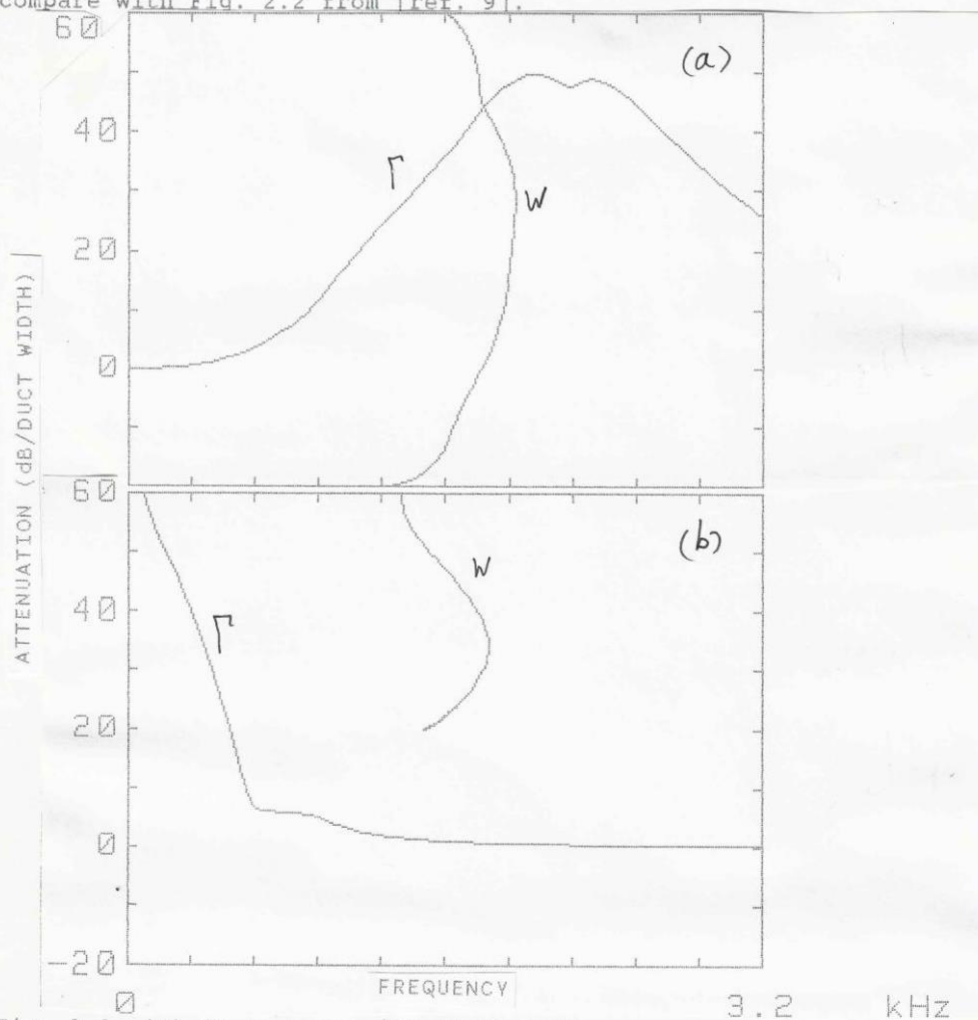


Fig. 3.4. (a) (0,0) mode; (b) (0,2) mode  
 $\Gamma$  attenuation;  $w$  Nyquist plot; Freq. res. 32 Hz  
 Duct width  $l = 0.5334$  m; lining thickness  $d = 0.0300$  m  
 $\sigma = 32,000$  N·s/m<sup>4</sup>.

Figure 3.4. Detail of Fig. 3.3(f): (0,0) and (0,2) modes with Nyquist diagrams.  $l = 0.5334$  m;  $d = 0.030$  m;  $\sigma = 32000$  N·s/m<sup>4</sup>.

Fig. 3.4 (a) was produced entirely with the Newton-Raphson method using altogether 235 iterations for 100 frequencies. Likewise Fig. 3.4 (b) was produced entirely with Muller's method using a total of 952 iterations for the 100 frequencies. No problems with either iteration were recorded. The Nyquist plots are smooth, without any sudden deviations. However the double peak in attenuation in Fig. 3.4 (a) is probably due to the presence of yet another higher order mode as is indicated by the high frequency end point value of  $w$ . The above also shows that it is the (0,2) mode that approaches  $\frac{1}{2}\pi + j\cdot 0$  in the limit. One can only conclude that in this case it is the (0,2) mode that is the least attenuated mode above cut on.

Fig. 3.5 shows the two modes (0,0) and (0,2) in Fig. 3.3 (h) and (g) now separated. A print out of the arrays for  $w$  has unfortunately been mislaid but the values were seen by the author to be approaching those below:

Fig. 3.5 (a) (0,0) mode;

$$f = 32 \text{ Hz}; \quad \text{Re}\{w\} = 0; \quad \text{Im}\{w\} = 0$$

$$f = 3.2 \text{ kHz}; \quad \text{Re}\{w\} = \frac{1}{2}\pi; \quad \text{Im}\{w\} = 0$$

Fig. 3.5 (b) (0,2) mode;

$$f = 32 \text{ Hz}; \quad \text{Re}\{w\} = \pi; \quad \text{Im}\{w\} = 0$$

$$f = 3.2 \text{ kHz}; \quad \text{Re}\{w\} = 1\frac{1}{2}\pi; \quad \text{Im}\{w\} = 0$$

No difficulty was encountered with the Newton-Raphson method employed to generate curve (a). The iteration in curve (b) encountered difficulties at 886.60 Hz and from 894.66 Hz the iteration progressed smoothly after switching over to using the Newton-Raphson method. These frequencies correspond to the position in Fig. 3.3 (h) where with the lower frequency resolution of 32 Hz the (0,2) mode attenuation curve followed that for the (0,0) mode. Again it should be noticed that these frequencies are around cut on for the (0,2) mode. The other figures in Fig. 3.3 continue to show the two modes (0,0) and (0,2) as quite separate. No further problems were encountered with the iterations and the low and high frequency values for  $w$  remained throughout as those shown above for Fig. 3.5.

It is interesting to note that in this example, increasing the lining thickness beyond 0.1 m has no noticeable effect on the attenuation curves for either the (0,0) or the (0,2) mode.

Neither of the modes shown in Fig. 3.6 is the least attenuated. The Nyquist plot in Fig. 3.6 (b) shows convergence to a higher root, probably (0,4). There is no doubt that again in this example that the (0,2) mode is the least attenuated above cut on. As explained in [ref. 9] and mentioned on p. 14, the least attenuated mode has a high frequency end point for  $w$  at  $\frac{1}{2}\pi + j\cdot 0$ . Only the (0,0) and (0,2) modes can have this value for  $w$  as the

Figure 3.5. Modes (0,0) and (0,2) from Fig. 3.3(h) separated.  $l = 0.5334 \text{ m}$ ;  $d = 0.040 \text{ m}$ ;  $\sigma = 32000 \text{ N}\cdot\text{s}/\text{m}^2$ ;  $F_{\text{res}} = 8.06 \text{ Hz}$ .

high frequency end point. An investigation of the behaviour of the (0,0) mode in Fig. 3.6 (a) does reveal that  $w$  appears to increase without limit. Compare the Nyquist plot in Fig. 3.6 (a) with the Nyquist plot for the (0,0) mode shown in Fig. 2.2.

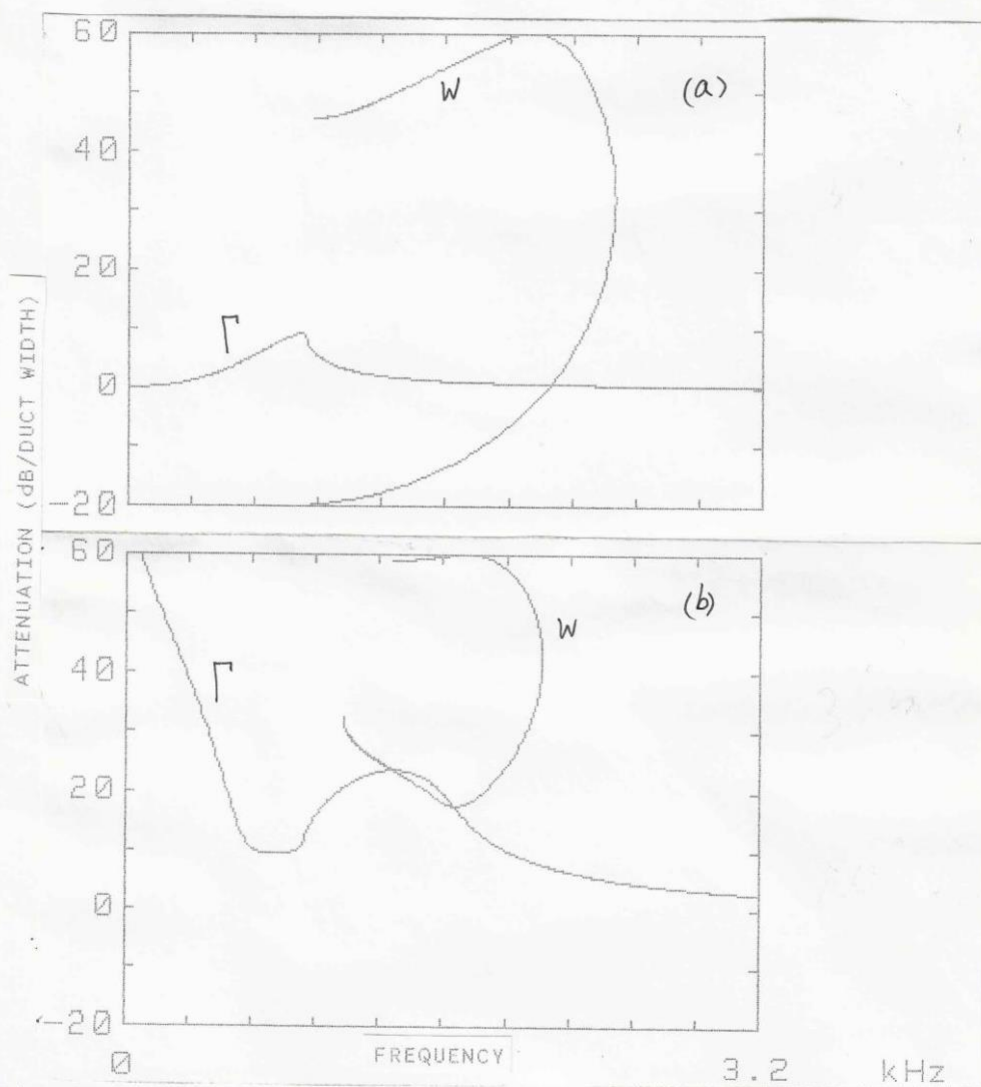


Fig. 3.5. (a) (0,0) mode; (b) (0,2) mode  
 $\Gamma$  attenuation;  $w$  Nyquist plot; Freq. res. 8.06 Hz  
 Duct width  $l = 0.5334$  m; lining thickness  $d = 0.0400$  m  
 $\sigma = 32,000$  N·s/m

Figure 3.6.  $l = 0.5334$  m;  $d = 0.1524$  m;  $\sigma = 2000$  N·s/m<sup>4</sup> (curve no. 4 in Fig. 3.2).

Fig. 3.7 shows that with an increase in  $\sigma$  from 2000 N·s/m to 3000 N·s/m then the (0,0) mode is once again the least attenuated mode. It starts with  $w$  at  $0 + j \cdot 0$  and possess a high frequency end point of  $\frac{1}{2}\pi + j \cdot 0$ . The 'peculiar' spiked peak (see

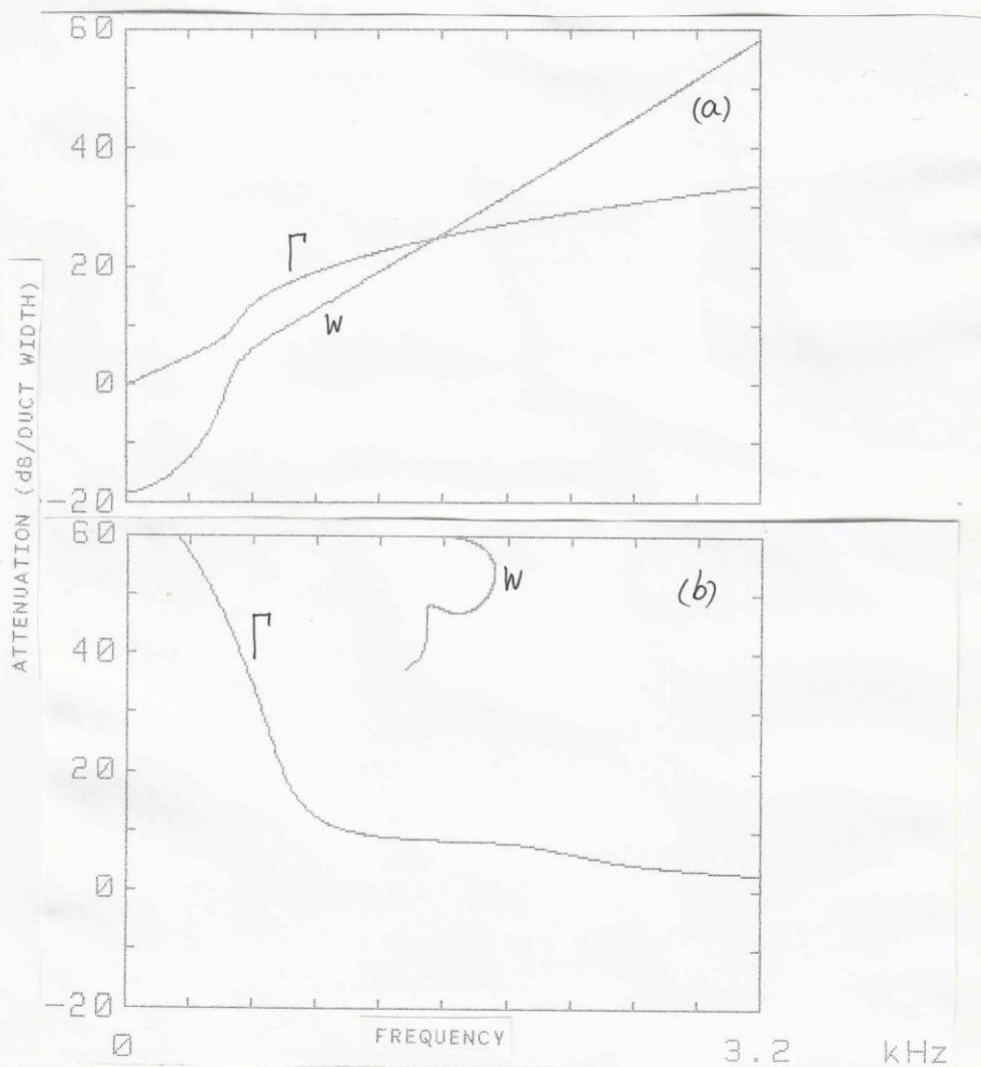


Fig. 3.6. (a) (0,0) mode; (b) (0,2) mode  
 $\Gamma$  attenuation;  $w$  Nyquist plot; Freq. res. 8.06 Hz  
 Duct width  $l = 0.5334$  m; lining thickness  $d = 0.1524$  m  
 $\sigma = 2,000$  N·s/m . Compare with Fig. 3.2 curve no. 4.

Figure 3.7.  $l = 0.5334$  m;  $d = 0.1524$  m;  $\sigma = 3000$  N·s/m<sup>4</sup> (curve no. 5 in Fig. 3.2).

again curve 5 in Fig. 3.2) and the wobble in the Nyquist plot does lead the author to question the validity of this result. It is still possible that the (0,0) calculation has got mixed up with the 'true' roots for the (0,2) mode.

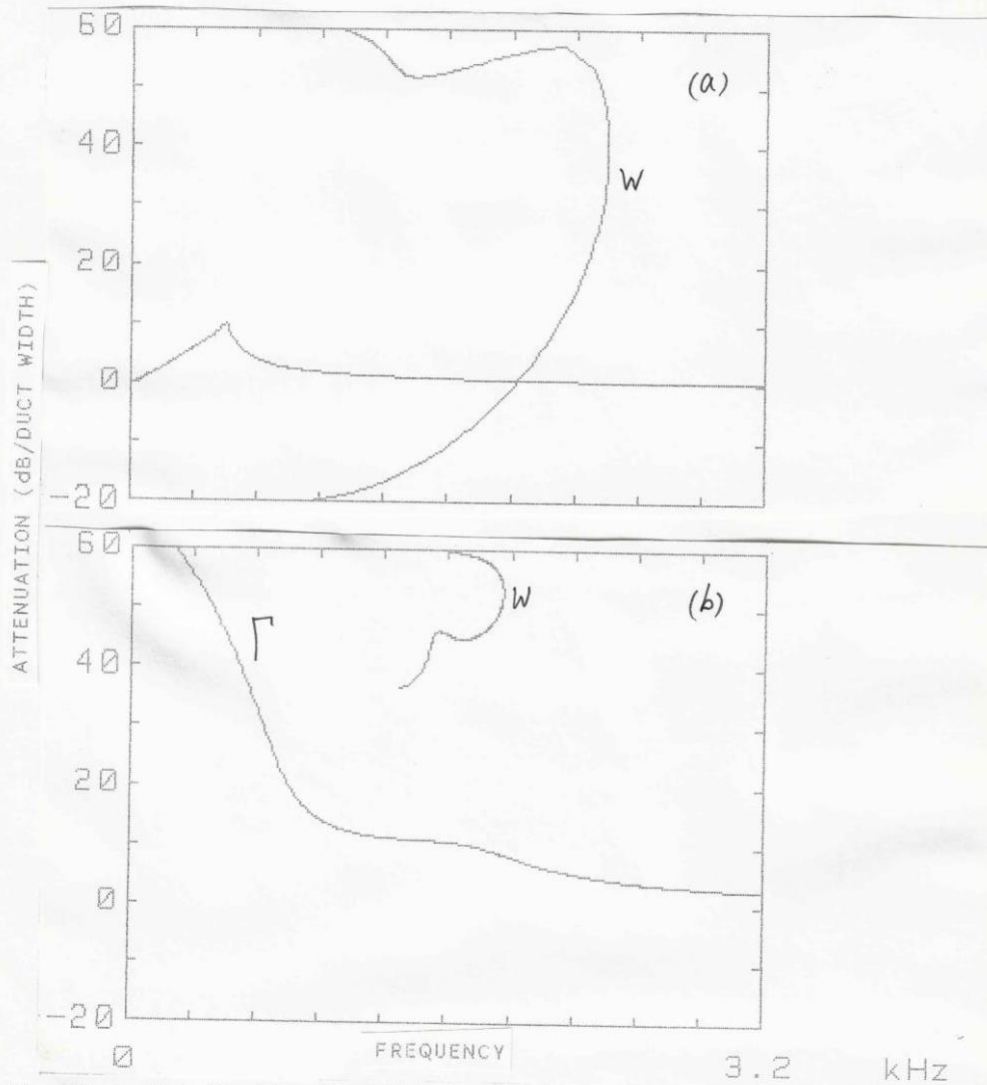


Fig. 3.7. (a) (0,0) mode; (b) (0,2) mode  
 $\Gamma$  attenuation; w Nyquist plot; Freq. res. 8.06 Hz  
 Duct width  $l = 0.5334$  m; lining thickness  $d = 0.1524$  m  
 $\sigma = 3,000$  N·s/m . Compare with Fig. 3.2 curve no. 5.

Figure 3.8.  $l = 0.5334$  m;  $d = 0.1524$  m;  $\sigma = 128000$  N·s/m<sup>4</sup> (curve no. 11 in Fig. 3.2).

Please note that Fig. 3.8 has a logarithmic frequency scale. In this figure it can be seen that neither the (0,0) nor the (0,2) mode have smooth Nyquist plots. Difficulty in converging was experienced in both plots at 1552 Hz and after this frequency the two run together. At this frequency Newton-Raphson's method for the (0,0) mode required 10 iterations and Muller's method for the (0,2) mode required 101 iterations. Apart from experiencing

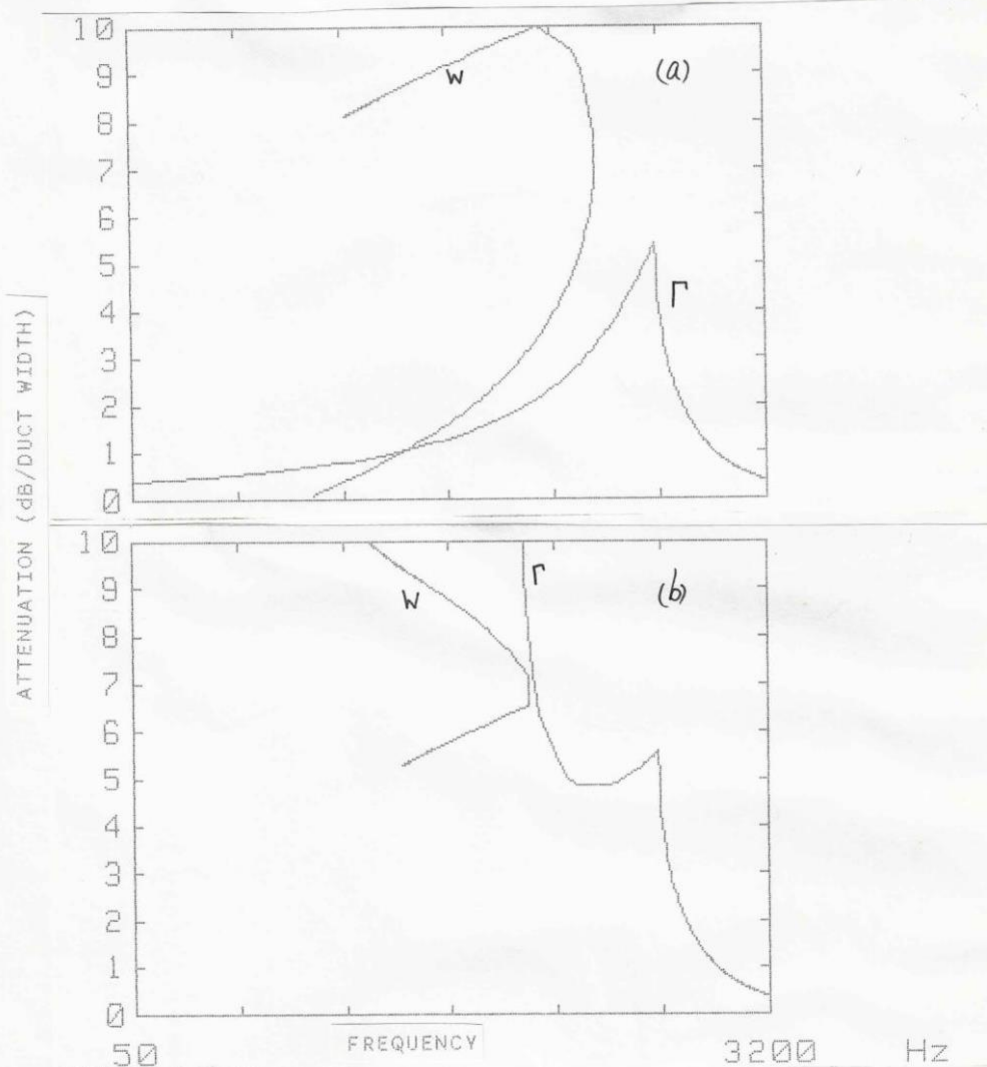


Fig. 3.8. (a) (0,0) mode; (b) (0,2) mode  
 $\Gamma$  attenuation; w Nyquist plot; Freq. res. 16 Hz  
 Duct width  $l = 0.5334$  m; lining thickness  $d = 0.1524$  m  
 $\sigma = 128,000$  N's/m . Compare with Fig. 3.2 curve 11.

Figure 3.9.  $l = 0.5334$  m;  $d = 0.1524$  m;  $\sigma = 256000$  N's/m<sup>4</sup> (curve no. 12 in Fig. 3.2).

The plot in Fig. 3.8 (a) is undoubtedly the 'curve' of least attenuation, it starts at  $0 + j \cdot 0$  and is approaching  $\frac{1}{2}\pi + j \cdot 0$  at 3.2 kHz. Whether this curve consists purely of the (0,0) mode or it continues along the (0,2) mode after 1552 Hz is not possible yet for the author to comment on. The same caution should be shown here as is expressed on p. 19 concerning Fig. 2.4 (b).

Fig. 3.9 displays Nyquist diagrams that are smooth and a closer investigation of their behaviour reveals that  $w$  approaches the values shown below:

Fig. 3.9 (a) (0,0) mode;

$$\begin{aligned} f &= 8.06 \text{ Hz}; & \operatorname{Re}\{w\} &= 0; & \operatorname{Im}\{w\} &= 0 \\ f &= 3.2 \text{ kHz}; & \operatorname{Re}\{w\} &= \frac{1}{2}\pi \text{ (increasing)}; \\ & & \operatorname{Im}\{w\} &= \pi \text{ (increasing)}; \end{aligned}$$

Fig. 3.9 (b) (0,2) mode;

$$\begin{aligned} f &= 8.06 \text{ Hz}; & \operatorname{Re}\{w\} &= \pi; & \operatorname{Im}\{w\} &= 0 \\ f &= 3.2 \text{ kHz}; & \operatorname{Re}\{w\} &= \frac{1}{2}\pi; & \operatorname{Im}\{w\} &= 0 \end{aligned}$$

No problems were encountered with convergence which ran smoothly and rapidly in both cases. Using Newton-Raphson's method, 858 iterations were used to produce Fig. 3.9 (a) and 4171 iterations were used to produce Fig. 3.9 (b) with Muller's method. In both cases the attenuation was calculated for 397 evenly spaced frequencies in the interval 0 Hz - 3.2 kHz, hence the frequency resolution of 8.06 Hz. It seems that again it is the (0,2) mode that is the least attenuated above cut on.

Finally, another example of this 'anomalous behaviour' is taken from Wassilieff, Fig. 4. (a) in [ref. 9] reproduced here as Fig. 3.10 (a) for comparison with Fig. 3.10 (b).

Wassilieff maintains that the first part of the attenuation curve (to the left of the peak) is due to the (0,0) mode and that the remainder (to the right of the peak) is due to the (0,2) mode. The author has not yet found in any of the published literature attenuation curves that show the modes separately, such as those presented in this thesis.

Applying the same techniques as those employed for the parameter study in this subsection the author has been able to separate the (0,0) mode from the (0,2) mode and thereby verify the observation made by Wassilieff, compare figures (a) and (b).

Fig. 3.10 (a) is shown in more detail in Fig. 3.11 (a) and (b). Note though the logarithmic scale and compare with Fig. 3.9.

Please note that the dots in Fig. 3.10 (a) are measurements. In view of the excellent agreement between figures (a) and (b) these measurements serve also to verify the algorithm 'Extended'. The author's own verification is the subject of the next section, § 4.

Figure 3.10. (a) Reproduced from [Ref. 9, Fig. 4a]. (b) Produced using 'Extended':  $\sigma = 8500 \text{ N}\cdot\text{s}/\text{m}^4$ ,  $l = 0.5 \text{ m}$ ,  $d = 0.05 \text{ m}$ ,  $F_{\text{res}} = 20.0 \text{ Hz}$ .

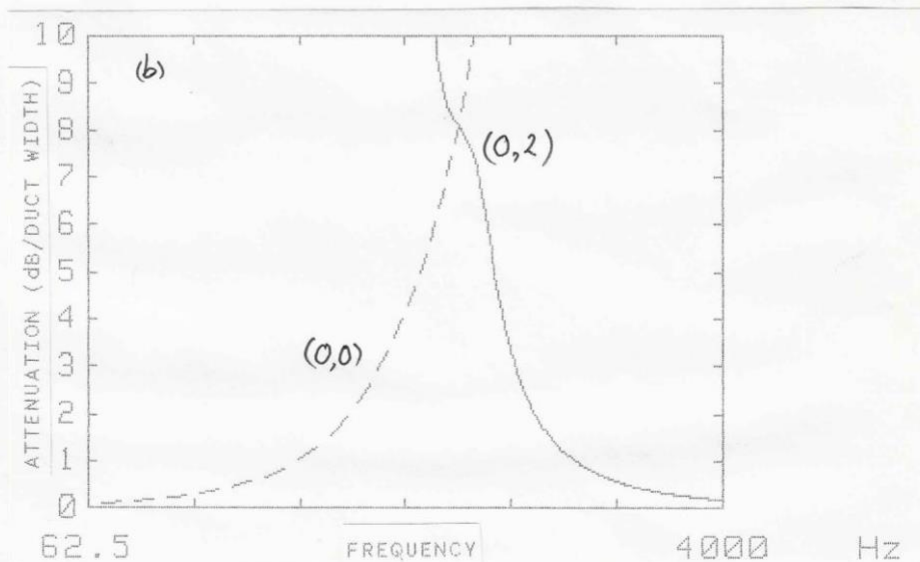
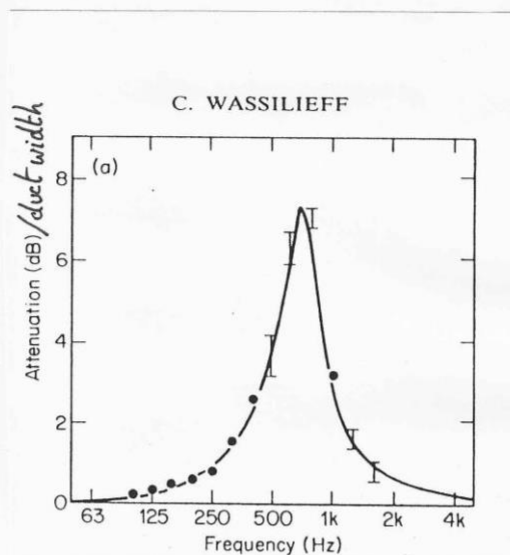


Fig. 3.10. (a) Reproduced from Fig. 4. (a) [ref. 9].  
 (b) Produced using 'Extended'  
 $\Gamma$  attenuation; w Nyquist plot  
 Freq. res. 20.0 Hz;  $\sigma = 8,500 \text{ N}\cdot\text{s/m}$ .  
 Duct width  $l = 0.5 \text{ m}$ ; lining thickness  $d = 0.05 \text{ m}$

Figure 3.11. Detail of Fig. 3.10, showing (0,0) and (0,2) modes separately.

### 3.5 Programme Printouts: 'Material' and 'Extended'

The following listings are reproduced as they appear in the original programme.

## Subroutine 'Material'

```
7910 !                      Material (properties of absorbing material)
7920 SUB Material
7940 !      Based on empirical formulae of Delany and Bazley as modified
by
7950 !      F. P. Mechel: ACUSTICA vol.35 (1976) p. 210-213.
7970      COM /Data/ REAL
Zd(*),Zs,L(*),Ltot,R(*),Ri(*),Ro(*),W(*),Wo(*),Wi(*),H(*)
7980      COM /Tally/ INTEGER I,J,N,Max
7990      COM /Abs/ REAL
A(*),Zr(*),Zi(*),Gamr(*),Gami(*),Gr(*),Gi(*),Wre(*),Wim(*),T(*)
8010      DISP "COMPUTING PROPAGATION CONSTANT OF ABSORBING MATERIAL"
8030      K1=-1.4664
8040      K2=.2115
8050      K4=1.4035
8060      FOR J=1 TO Max
8070          A(J)=Af/J                      ! A = NORMALISED FLOW RESISTIVITY
8100          K=J*Kf
8110          IF A(J)<60 THEN
8120              Log=LOG(A(J))
8130              Zr(J)=1+.04885*EXP(.754*Log)      ! LINING CHARAC.
IMPEDANCE
8140              Zi(J)=-.087*EXP(.7307*Log)
8150              Gamr(J)=K*.189*EXP(.6185*Log)      ! LINING PROPAGATION
CONST.
8160              Gami(J)=K*(1+.0978*EXP(.6929*Log))
8170          ELSE
8190              CALL Csqr(K1,K5,D1,D2)
8200              CALL Cdiv(K3,K4,D1,D2,E1,E2)
8220              Zr(J)=E1                      ! LINING CHARAC.
IMPEDANCE
8230              Zi(J)=E2
8240              Gamr(J)=K*D1                      ! LINING PROPAGATION
CONST.
8250              Gami(J)=K*D2
8260          END IF
8270      NEXT J
8280 SUBEND
```

## Subroutine 'Extended' (Extended Reaction)

```
8880 SUB Extended(Fr$,S$)
8960      B=2*Ri(N)/X(N)                      ! b IN SCOTT/CHRISTIE EQUATION
8970      L1=X(N)/2                          ! half the duct width (air space)
8980      K3=1                              ! branch selector for arctan
8990      K4=0                              ! mode flag (0 = (0,0) mode)
9010      INPUT "WAVE TYPE (0,0) MODE (0) OR (0,2) MODE (1) (default
(0)) ?:" ,K4
9070      K5=0                              ! jump counter flag
9080      Ep=1.0E-15                        ! ERROR BOUND (Newton-Raphson)
9120      FOR J=1 TO Max
9150          K=Kf*J                          ! wave number
9170          CALL Cmult(Gamr(J),Gami(J),Gamr(J),Gami(J),D1,D2)      ! Gam^2
(material)
9190          CALL Cmult(Gr(J),Gi(J),Gr(J),Gi(J),D3,D4)              !
Gamma^2 (duct)
9210          CALL Cmult(Zr(J),Zi(J),Gamr(J),Gami(J),E1,E2)          !
(Z/rho/C)*Gam
9230          CALL Cdiv(0,K,E1,E2,M1,M2)      ! M = jK*rho*C/(Z*Gam)
9250          W3=(D3+(K*K))*L1*L1              ! Re{w*w}
```

```

9270      W4=(D4*L1*L1)      ! Im{w*w}
9290      P3=-(D1+(K*K))*L1*L1      ! Re{p*p}
9310      P4=-(D2*L1*L1)      ! Im{p*p}
9330      CALL Csqqr(W3+P3,W4+P4,X1,X2)      ! sqrt(w^2+p^2)
9350      CALL Ctan(B*X1,B*X2,X3,X4)      ! tan(b*sqrt(w^2+p^2))
9430      IF K4=1 THEN GOTO 10290      ! Switch to Muller for
(0,2) mode
9870      IF ABS(F7)>Ep OR ABS(F8)>Ep THEN GOTO 9910
9910      W1=W1-F5
9930      W2=W2-F6
9970      T(J)=T(J)+1
9990      IF T(J)>100 THEN GOTO 10030
10030     Wre(J)=W1
10050     Wim(J)=W2
10110     E1=ABS(Wre(J-1)-Wre(J-2))
10130     E2=ABS(Wim(J-1)-Wim(J-2))
10150     E3=ABS(Wre(J)-Wre(J-1))
10170     E4=ABS(Wim(J)-Wim(J-1))
10190     IF SQR((E1*E1)+(E2*E2))>SQR((E3*E3)+(E4*E4))/3 THEN GOTO 10840
10210     K5=K5+1      ! JUMP HAS OCCURRED
10290     CALL Cdiv(-X7,-X8,W1,W2,Y1,Y2)
10430     Y3=(K3*PI)+((J4-J2)/2)
10450     Y4=(1/2)*(LOG(J1/J3))
10470     En=1.0E-10      ! ERROR BOUND (Muller)
10700     Wre(N)=2*Wre(N-1)-Wre(N-2)      ! LINEAR EXTRAPOLATION
10710     Wim(N)=2*Wim(N-1)-Wim(N-2)
10840     CALL Csqqr((W3/L1/L1)-(K*K),W4/L1/L1,D5,D6)      ! Gamma
10860     Gr(J)=ABS(D5)      ! DUCT PROPAGATION CONSTANT
10880     Gi(J)=ABS(D6)
10990     IF Fr$="2" THEN Gr(N)=2*Gr(N)      ! Square duct lined all around
11020     IF S$="C" THEN Gr(N)=2*Gr(N)      ! Circular duct
11060 SUBEND

```

## 4. Measurements

### 4.1 Insertion Loss

In order to verify the algorithm 'Extended', insertion loss (IL) measurements were performed on the silencer shown in Figure 4.1. IL is defined as the difference between the acoustic power without any filter and that with the filter [Ref. 14; § 2.10.1]. A reference duct of the same length and cross-section was used to ensure constant radiation impedance throughout.



Fig. 4.1 (a) The experimental set up.

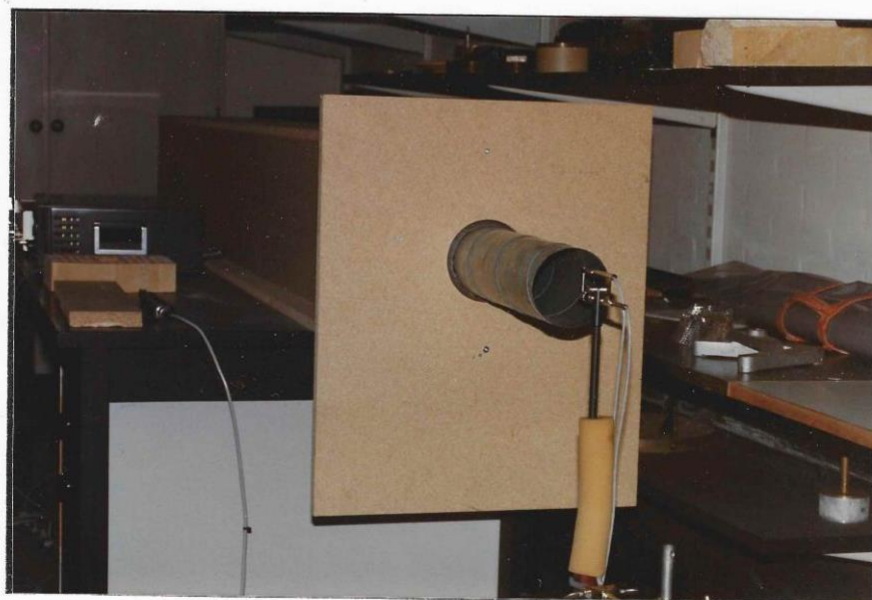


Fig. 4.1 (b) Positioning the intensity probe.

*Figure 4.1. (a) The experimental set-up. (b) Positioning the intensity probe.*



Fig. 4.1 (c) The silencer. Notice the structure of the lining.

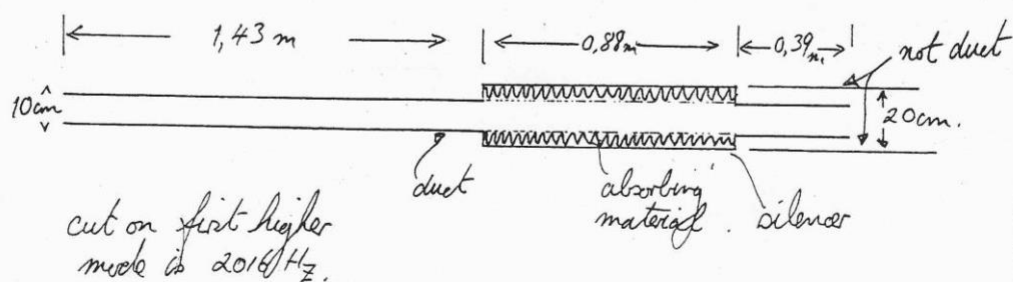


Fig. 4.2 The silencer in place in the duct (schematic).

Figure 4.1(c). The silencer. Below: Figure 4.2. Schematic of the silencer in the duct. Duct radius 5 cm; silencer length 0.88 m; lining outer radius 10 cm.

Measurements were made up to 3.2 kHz using a Brüel & Kjær type 2032 dual-channel FFT analyser. A 12 mm probe spacing gives an associated error no greater than  $-1$  dB in the range 100 Hz to 4 kHz [Ref. 21].

Three measurements were performed:

1. Reference system: straight duct, length 2.7 m, radius 5 cm.
2. Duct plus silencer with the cylindrical perforated plate.
3. Duct plus silencer without the cylindrical perforated plate.

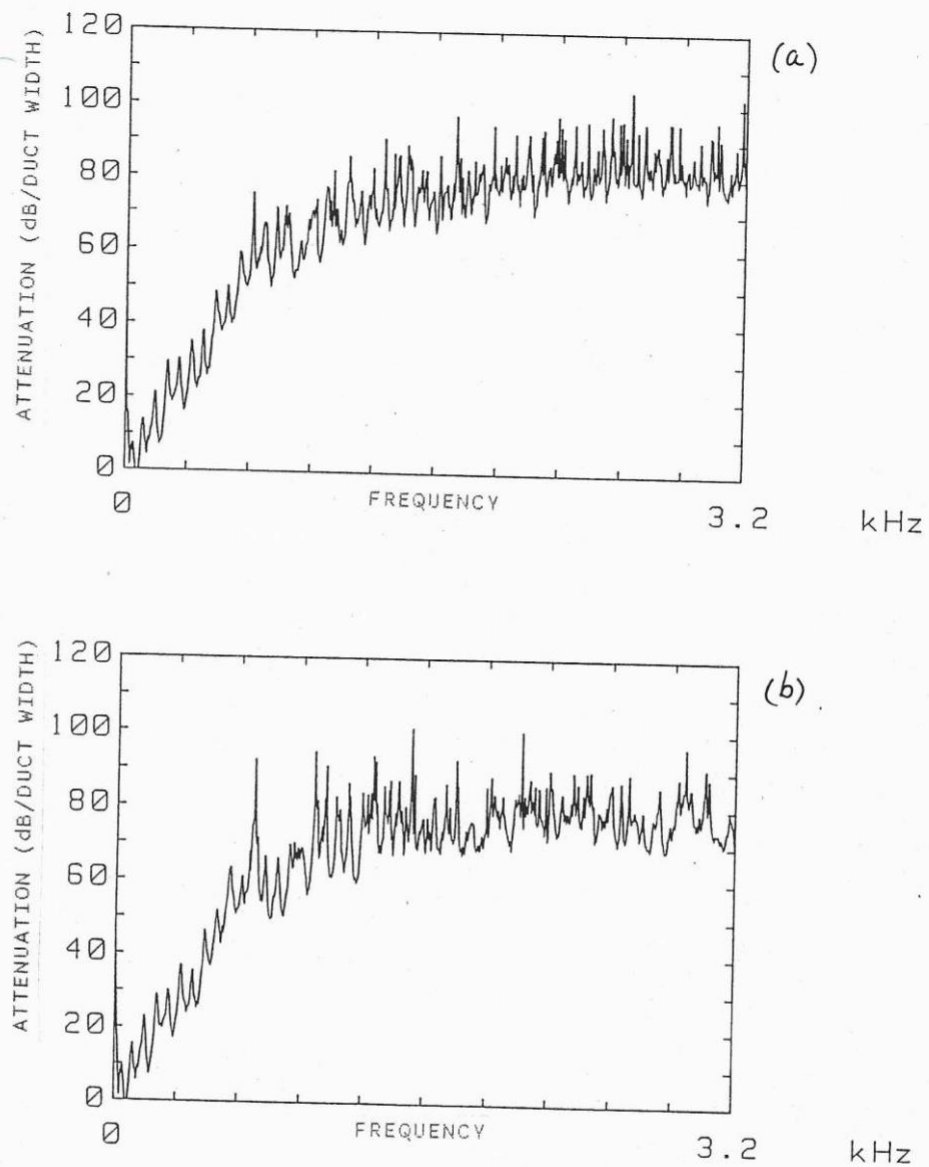


Fig. 4.4 The IL measured for the arrangement shown schematically in Fig. 4.2.  
 (a) without the perforated cylinder.  
 (b) with the perforated cylinder.

Figure 4.4. Insertion loss. (a) Without the perforated cylinder. (b) With the perforated cylinder. Attenuation is higher at high frequencies without the perforated cylinder, as predicted in § 1.

## 4.2 Flow Resistivity

Flow resistivity was calculated from dynamic measurements of flow resistance [Ref. 20]. The perforated cylinder support mesh had negligible flow resistance (Figure 4.3).

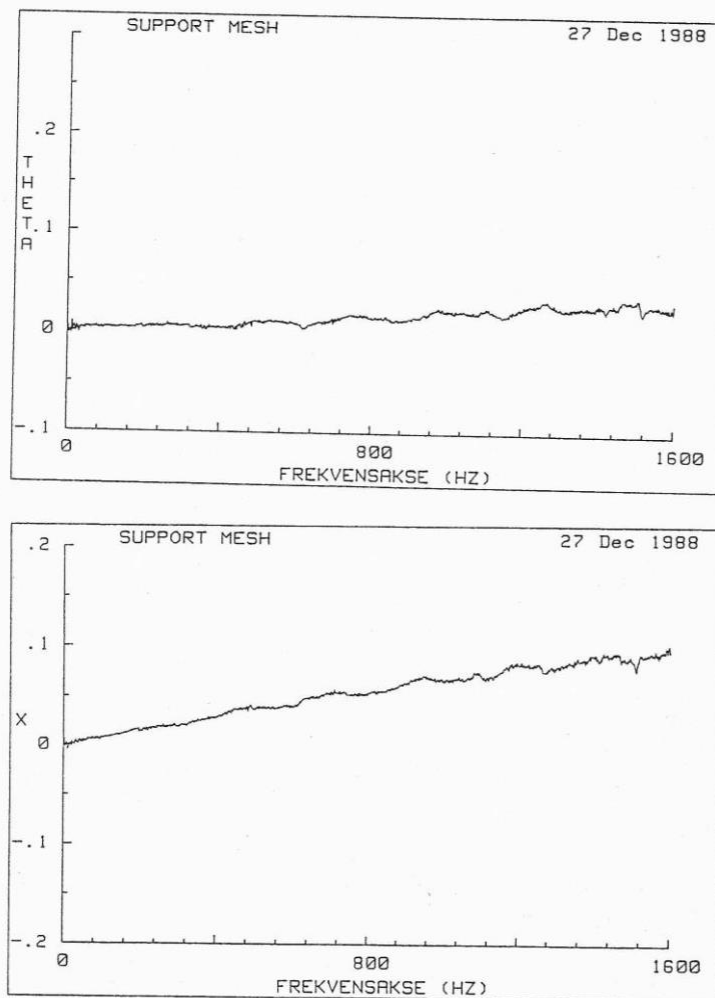


Fig. 4.3 The flow resistance of the perforated cylinder, used as a support mesh for the lining specimens. THETA is the real part of the flow resistance, X (Chi) is the imaginary part of the flow resistance.

Figure 4.3. Flow resistance of the perforated cylinder support mesh (27 Dec. 1988). THETA = real part; X (Chi) = imaginary part.

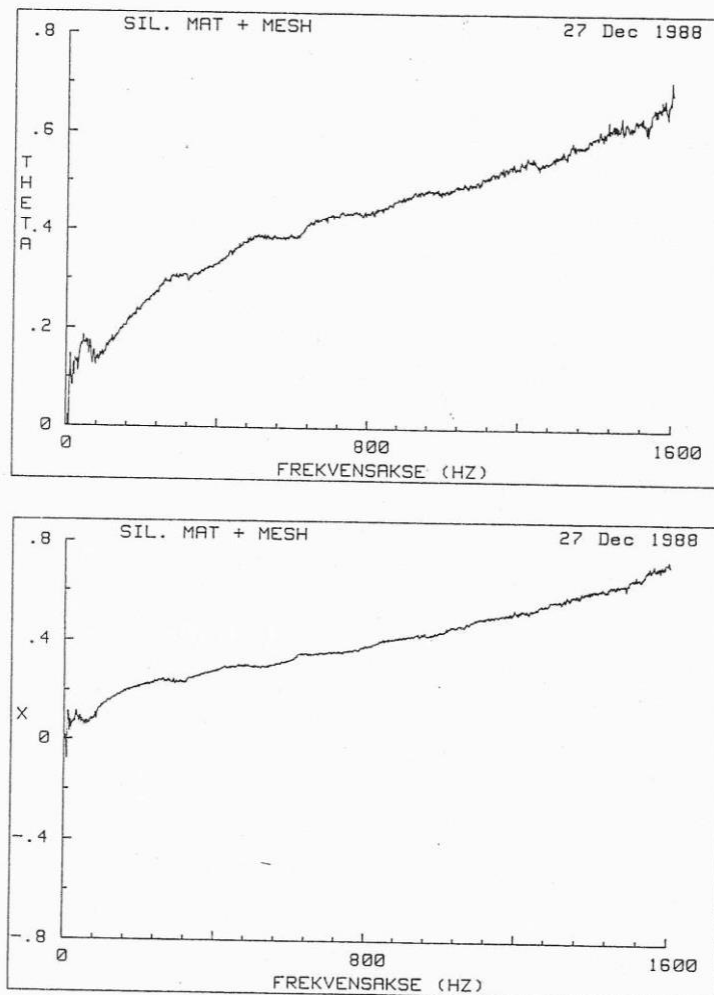


Fig. 4.5 The flow resistance of the lining together with the supporting mesh. THETA is the real part of the flow resistance, X (Chi) is the imaginary part of the flow resistance.

Figure 4.5. Flow resistance of the lining material with the supporting mesh (27 Dec. 1988). THETA = real part; X (Chi) = imaginary part.

From Figure 4.5(a), for a sample thickness of 0.75 cm at  $f = 1600$  Hz,  $\theta = 0.6$ , giving:

$$\sigma = \left( \frac{0.6 \times 415}{0.0075} \right) = 33\{, \}200 \text{ N} \cdot \text{s}/\text{m}^4$$

Accounting for thickness non-uniformity ( $\pm 2$  mm) and frequency dependence of  $\theta$ :

$$30\{, \}000 < \sigma < 40\{, \}000 \text{ N} \cdot \text{s}/\text{m}^4$$

### 4.3 Comparison with Calculations

The algorithm 'Extended' was used to calculate the IL for both bounding values of  $\sigma$ , using the dimensions in Figure 4.2. Results are shown in Figure 4.6.

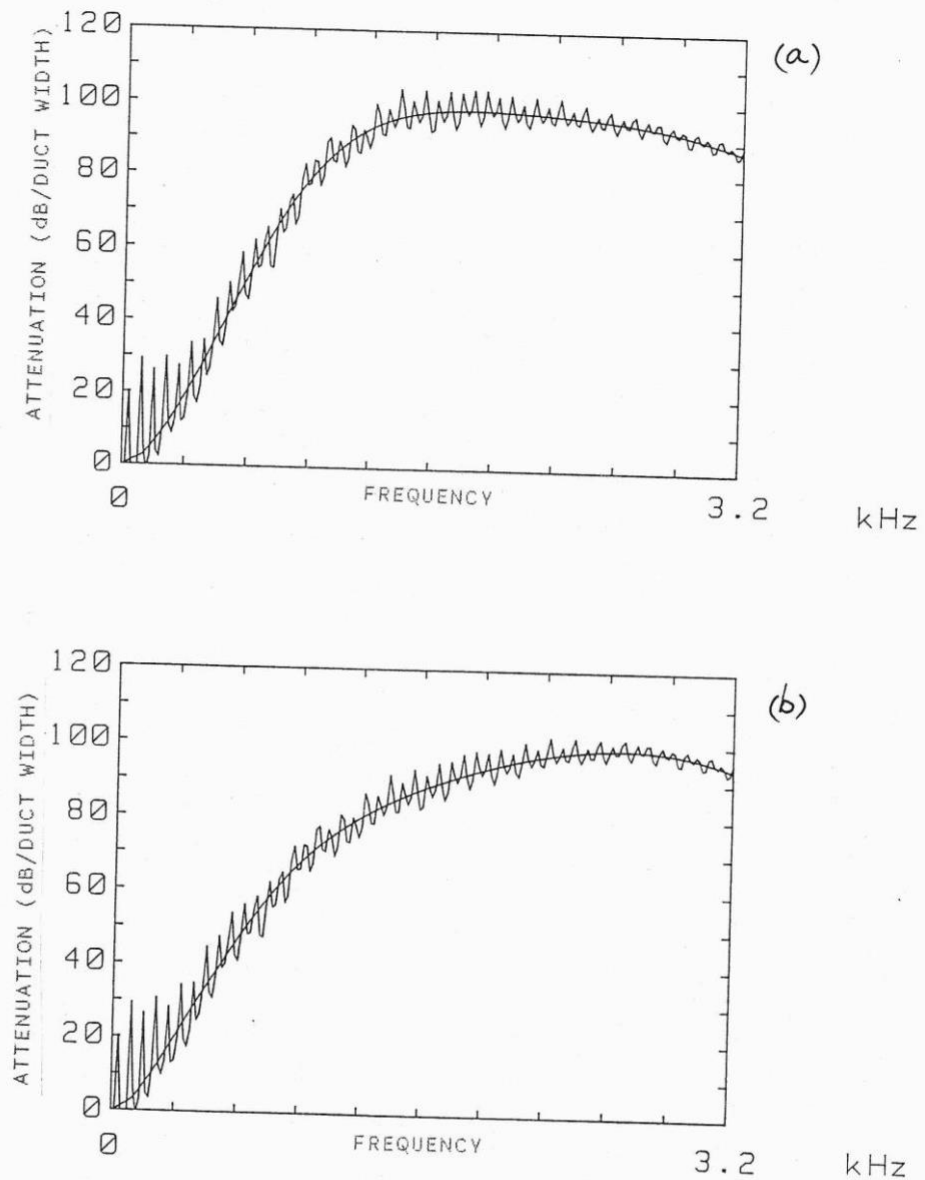


Fig. 4.6 IL calculated, using 'Extended', for the arrangement shown in Fig. 4.2.  
 (a)  $\sigma = 30000 \text{ N}\cdot\text{s}/\text{m}^4$ ; (b)  $\sigma = 40000 \text{ N}\cdot\text{s}/\text{m}^4$

Figure 4.6. IL calculated using 'Extended'. (a)  $\sigma = 30000 \text{ N}\cdot\text{s}/\text{m}^4$ ; (b)  $\sigma = 40000 \text{ N}\cdot\text{s}/\text{m}^4$ .

Calculations with 'Extended' are in much better agreement with experiment than those from Morse's theory. Remaining discrepancies are largely attributable to the

reactive sound field and the flexibility of the duct walls, which are not accounted for in Scott's theory.

## 5. Conclusion

One of the most striking features revealed in this study is the anomalous behaviour of the (0,0) and (0,2) modes for frequencies above cut-on for (0,2). The (0,0) mode is not always the least attenuated. This occurs only for certain ratios of lining thickness to duct width and for certain flow resistivities.

The idea of separate (0,0) and (0,2) modes, fully applicable for hard-walled ducts, is probably no longer fully applicable for soft-walled ducts. When the (0,0) mode is not the least attenuated, it is as if the sound energy is being ever more diffracted out towards the sides of the duct, so the wave front is no longer plane.

Comparison between calculations with 'Extended' and published literature [Refs. 5 & 9] is very good. The result in Figure 3.10 is in excellent agreement with Wassilieff's calculations and measurements.

Suggested improvements to the algorithm include: a more sophisticated definition of a 'jump' considering discontinuities in the first differential of both  $\Gamma$  and  $w$ ; and calculating two arrays simultaneously to produce a composite 'curve of least attenuation'.

To conclude: below cut-on for the (0,2) mode, the (0,0) mode is always the least attenuated. There exists at least one frequency region in which sound propagation in ducts with absorbent linings can be safely calculated.

*“Skønt er det, vi ser,  
skønnere det, vi erkender,  
skønnest af alt er det, vi ikke fatter.”*

Den Salige Niels Steensen (1638–86)  
Biskop og videnskabsmand

## List of Symbols

|               |                                                                         |
|---------------|-------------------------------------------------------------------------|
| $A$           | Normalised flow resistivity ( $= \sigma/\rho f$ )                       |
| $b$           | Ratio of lining thickness to duct half-width ( $= d/l$ )                |
| $c$           | Free-field speed of sound (344.25 m/s at 20°C)                          |
| $d$           | Lining thickness                                                        |
| $f$           | Frequency                                                               |
| $h$           | Propagation constant in lining; $h = -j\Gamma a$                        |
| $j$           | Square root of $-1$                                                     |
| $K$           | Bulk modulus                                                            |
| $k$           | Wave number ( $= \omega/c$ )                                            |
| $l$           | Duct half-width (air space)                                             |
| $M$           | Ratio of density of air $\rho$ to complex density $\rho'$ of the lining |
| $m, n$        | Zero or positive integers (mode indices)                                |
| $P$           | Atmospheric pressure; also porosity                                     |
| $p$           | Sound pressure                                                          |
| $R$           | Real part of characteristic impedance $Z_a$ of lining                   |
| $w$           | Complex variable defined in Equation 2.2.8                              |
| $Z_a$         | Characteristic impedance of lining                                      |
| $\alpha$      | Real part of propagation constant in [Ref. 9]                           |
| $\beta$       | Imaginary part of propagation constant in [Ref. 9]                      |
| $\Gamma$      | Common propagation constant (lining and duct)                           |
| $\Gamma a$    | Propagation constant of lining ( $= jh$ )                               |
| $\varepsilon$ | Error bound                                                             |
| $\sigma$      | Flow resistivity ( $\text{N}\cdot\text{s}/\text{m}^4$ )                 |
| $\Phi$        | Particle velocity potential (air)                                       |
| $\theta$      | Particle velocity potential (lining)                                    |

The meaning of other symbols is given where they occur.

# Appendix A

## Web recreation of Figure 3.10(b)

In April 2026, thirty-seven years after the thesis was written, the '*Extended*' algorithm was ported from the original HP-BASIC to plain JavaScript and used to re-render Figure 3.10(b) on an interactive web page. The page solves Scott's transcendental equation (2.2.9) at 20 Hz frequency resolution for both the (0,0) and (0,2) modes, and allows the reader to vary  $\sigma$ ,  $l$ ,  $d$  and the frequency range in real time.

The live page is available at:

<https://michaelhgrant.github.io/Natur-fysik-kemi-mat/thesis2.html>

### A.1 Porting notes

The JavaScript implementation is a faithful port of the subroutines `Material` and `Extended` from § 3.5. The physics is unchanged: the Delany–Bazley/Mechel formulae (eqs. 2.1.1–2.1.2) for the lining properties, Scott's equation (2.2.9) for the propagation constant, and Newton–Raphson iteration with linear extrapolation of the previous two solutions as the initial guess for the next frequency step — exactly as implemented in programme lines 10910–10920.

Three small points of implementation are worth recording:

4. Sign of  $M$ . The BASIC source on line 9230 computes  $M = +j \cdot k / (Z_a \cdot \Gamma_a)$ , whereas the typeset derivation on p.12 gives  $M = -j \cdot k / (Z_a \cdot \Gamma_a)$ . The BASIC sign is the one that produces the correct physical behaviour and was the value used in all figures; the typeset text contains a sign error.
5. Damped Newton–Raphson. The Newton step is clamped to  $|\Delta w| \leq 0.5$  to prevent the iteration from leaping across to another mode's valley. This makes the 'jump' test on programme line 10190 rarely fire and removes the need for the Muller fallback in typical parameter ranges.
6. Display range. Figure 3.10(b) as printed is clipped at 10 dB per duct width. Below cut-on for (0,2) the mode is evanescent and its attenuation is very large, so only the branch above cut-on (approx. 600 Hz for the reference case) appears inside the plot frame. Similarly, the (0,0) mode rises above 10 dB for  $f \gtrsim 800$  Hz and exits the frame.

With these three points in place, the JavaScript output agrees with the 1989 BASIC plot to the width of a printed line. The (0,0) mode rises smoothly from 0 dB at low frequency to about 8 dB at 700 Hz, then exits the frame; the (0,2) mode enters the frame at about 600 Hz at 9 dB, falls to a local minimum around 700 Hz, and decays to zero at high frequency. The composite 'least-attenuated envelope' — the curve  $\min\{\text{Re } \Gamma^{(0,0)}, \text{Re } \Gamma^{(0,2)}\}$  — reproduces the single peaked curve measured by Wassilieff and shown in Fig. 3.10(a).

### A.2 Source listing

The source of the web page is a single self-contained HTML file of approximately 1,000 lines, comprising HTML markup, CSS for a 1970s–style technical-plot aesthetic, and JavaScript implementing the complex-number arithmetic, the Delany–

Bazley/Mechel formulae, Scott's equation, damped Newton–Raphson with jump detection, and the frequency-sweep driver. The plot is rendered with Chart.js (CDN import). The complete source listing follows.

```
—— thesis2.html ———
<!DOCTYPE html>
<html lang="en">
<head>
<meta charset="UTF-8">
<meta name="viewport" content="width=device-width, initial-scale=1.0">
<title>Sound Propagation in Ducts with Absorbent Linings – Figure
3.10(b)</title>
<link rel="preconnect" href="https://fonts.googleapis.com">
<link rel="preconnect" href="https://fonts.gstatic.com" crossorigin>
<link
href="https://fonts.googleapis.com/css2?family=EB+Garamond:ital,wght@0,400;0,500;0,600;1,400&family=JetBrains+Mono:wght@400;500;700&display=swap"
rel="stylesheet">
<script
src="https://cdnjs.cloudflare.com/ajax/libs/Chart.js/4.4.1/chart.umd.min.js"></script>
<style>
:root {
  --paper: #f4efe4;
  --paper-dark: #e8e0d0;
  --ink: #1a1815;
  --ink-soft: #4a4640;
  --rule: #2a2620;
  --signal-red: #b23a2d;
  --signal-blue: #1e4b7c;
  --accent: #8a6c3a;
  --grid: rgba(42, 38, 32, 0.12);
  --grid-major: rgba(42, 38, 32, 0.25);
}
* { box-sizing: border-box; margin: 0; padding: 0; }
html { scroll-behavior: smooth; }
body {
  font-family: 'EB Garamond', Georgia, serif;
  background: var(--paper);
  background-image:
    radial-gradient(circle at 20% 10%, rgba(138, 108, 58, 0.06) 0%,
transparent 50%),
    radial-gradient(circle at 80% 90%, rgba(30, 75, 124, 0.04) 0%,
transparent 50%),
    repeating-linear-gradient(0deg, transparent 0, transparent 2px,
rgba(42, 38, 32, 0.015) 2px, rgba(42, 38, 32, 0.015) 3px);
  color: var(--ink);
  line-height: 1.6;
  min-height: 100vh;
  padding: 2rem 1.5rem;
}
.container { max-width: 1100px; margin: 0 auto; }

/* —— HEADER —— */
header {
  border-top: 3px double var(--rule);
  border-bottom: 1px solid var(--rule);
  padding: 2rem 0 1.5rem;
```

```

    margin-bottom: 2.5rem;
    position: relative;
}
header::before {
    content: "";
    position: absolute;
    top: -3px;
    left: 0;
    right: 0;
    height: 1px;
    background: var(--rule);
    transform: translateY(-5px);
}
.eyebrow {
    font-family: 'JetBrains Mono', monospace;
    font-size: 0.72rem;
    text-transform: uppercase;
    letter-spacing: 0.25em;
    color: var(--ink-soft);
    display: flex;
    justify-content: space-between;
    flex-wrap: wrap;
    gap: 1rem;
    margin-bottom: 1.2rem;
    padding-bottom: 0.8rem;
    border-bottom: 1px dotted var(--rule);
}
h1 {
    font-size: clamp(1.8rem, 4.5vw, 3rem);
    font-weight: 500;
    font-style: italic;
    letter-spacing: -0.01em;
    line-height: 1.1;
    margin-bottom: 0.5rem;
}
h1 small {
    display: block;
    font-style: normal;
    font-size: 0.5em;
    letter-spacing: 0.05em;
    color: var(--ink-soft);
    margin-top: 0.3rem;
    font-weight: 400;
}
.byline {
    font-family: 'JetBrains Mono', monospace;
    font-size: 0.8rem;
    color: var(--ink-soft);
    margin-top: 1rem;
    display: flex;
    gap: 2rem;
    flex-wrap: wrap;
}
.byline span::before {
    content: "► ";
    color: var(--accent);
}

/* — ABSTRACT — */
.abstract {
    background: var(--paper-dark);

```

```

padding: 1.5rem 2rem;
margin-bottom: 2.5rem;
border-left: 3px solid var(--signal-red);
font-size: 1.05rem;
position: relative;
}
.abstract-label {
font-family: 'JetBrains Mono', monospace;
font-size: 0.7rem;
text-transform: uppercase;
letter-spacing: 0.2em;
color: var(--ink-soft);
margin-bottom: 0.5rem;
}

/* — SECTION TITLES — */
h2 {
font-family: 'JetBrains Mono', monospace;
font-size: 0.95rem;
font-weight: 700;
text-transform: uppercase;
letter-spacing: 0.15em;
color: var(--ink);
margin: 3rem 0 1.2rem;
padding-bottom: 0.4rem;
border-bottom: 1px solid var(--rule);
display: flex;
align-items: baseline;
gap: 1rem;
}
h2::before {
content: attr(data-num);
color: var(--signal-red);
font-size: 0.85em;
}

/* — THE PLOT — */
.plot-card {
background: #fffcf5;
border: 1px solid var(--rule);
box-shadow:
0 1px 0 var(--paper-dark),
0 2px 4px rgba(26, 24, 21, 0.06),
0 8px 20px rgba(26, 24, 21, 0.04);
padding: 1.5rem 1.5rem 1rem;
margin-bottom: 2rem;
position: relative;
}
.plot-card::before {
content: "";
position: absolute;
top: 8px; left: 8px; right: 8px; bottom: 8px;
border: 1px solid var(--grid-major);
pointer-events: none;
}
.plot-header {
font-family: 'JetBrains Mono', monospace;
font-size: 0.7rem;
text-transform: uppercase;
letter-spacing: 0.18em;
color: var(--ink-soft);

```

```

    display: flex;
    justify-content: space-between;
    padding: 0.4rem 0.6rem 1rem;
    border-bottom: 1px dashed var(--grid-major);
    margin-bottom: 1rem;
}
.plot-header .fig-id { color: var(--signal-red); font-weight: 700; }
#plot-container {
    position: relative;
    height: 520px;
    width: 100%;
    padding: 0.5rem;
}
.plot-caption {
    font-size: 0.92rem;
    font-style: italic;
    color: var(--ink-soft);
    padding: 1rem 1rem 0.3rem;
    text-align: center;
    border-top: 1px dashed var(--grid-major);
    margin-top: 0.5rem;
    line-height: 1.5;
}
.plot-caption strong { font-style: normal; font-weight: 600; color:
var(--ink); }

/* — CONTROLS — */
.controls {
    display: grid;
    grid-template-columns: repeat(auto-fit, minmax(240px, 1fr));
    gap: 1rem;
    margin-bottom: 2rem;
    padding: 1.5rem;
    background: var(--paper-dark);
    border: 1px solid var(--rule);
}
.control {
    display: flex;
    flex-direction: column;
    gap: 0.5rem;
}
.control label {
    font-family: 'JetBrains Mono', monospace;
    font-size: 0.7rem;
    text-transform: uppercase;
    letter-spacing: 0.15em;
    color: var(--ink-soft);
}
.control .value {
    font-family: 'JetBrains Mono', monospace;
    font-size: 1rem;
    font-weight: 700;
    color: var(--signal-red);
}
.control input[type="range"] {
    -webkit-appearance: none;
    width: 100%;
    height: 2px;
    background: var(--rule);
    outline: none;
    margin: 0.5rem 0;
}

```

```

}
.control input[type="range"]::-webkit-slider-thumb {
  -webkit-appearance: none;
  appearance: none;
  width: 16px;
  height: 16px;
  background: var(--ink);
  border: 2px solid var(--paper);
  border-radius: 50%;
  cursor: pointer;
  box-shadow: 0 0 0 1px var(--ink);
}
.control input[type="range"]::-moz-range-thumb {
  width: 14px;
  height: 14px;
  background: var(--ink);
  border: 2px solid var(--paper);
  border-radius: 50%;
  cursor: pointer;
  box-shadow: 0 0 0 1px var(--ink);
}
.btn-reset {
  grid-column: -2 / -1;
  align-self: end;
  padding: 0.6rem 1rem;
  background: var(--ink);
  color: var(--paper);
  border: none;
  font-family: 'JetBrains Mono', monospace;
  font-size: 0.75rem;
  text-transform: uppercase;
  letter-spacing: 0.15em;
  cursor: pointer;
  transition: background 0.15s;
}
.btn-reset:hover { background: var(--signal-red); }

/* ——— LEGEND ——— */
.legend {
  display: flex;
  gap: 2rem;
  justify-content: center;
  padding: 0.8rem;
  font-family: 'JetBrains Mono', monospace;
  font-size: 0.78rem;
  border-top: 1px dashed var(--grid-major);
  flex-wrap: wrap;
}
.legend-item { display: flex; align-items: center; gap: 0.5rem;
cursor: pointer; user-select: none; }
.legend-item.disabled { opacity: 0.3; }
.legend-swatch {
  display: inline-block;
  width: 28px;
  height: 2px;
  position: relative;
}
.legend-swatch.dashed {
  background: transparent;
  background-image: linear-gradient(to right, var(--signal-red) 50%,
transparent 50%);

```

```

    background-size: 6px 2px;
    background-repeat: repeat-x;
}
.legend-swatch.solid.blue { background: var(--signal-blue); }
.legend-swatch.solid.black { background: var(--ink); height: 3px; }

/* — THEORY BLOCK ————— */
.theory {
    background: #fffcf5;
    border: 1px solid var(--rule);
    padding: 1.5rem 2rem;
    margin-bottom: 2rem;
    font-size: 0.98rem;
}
.theory p { margin-bottom: 1rem; }
.theory p:last-child { margin-bottom: 0; }
.eq {
    display: block;
    text-align: center;
    margin: 1rem 0;
    font-family: 'EB Garamond', serif;
    font-style: italic;
    font-size: 1.05rem;
}
.eq-num { font-style: normal; float: right; color: var(--ink-soft);
font-family: 'JetBrains Mono', monospace; font-size: 0.85em; }
code {
    font-family: 'JetBrains Mono', monospace;
    font-size: 0.88em;
    background: var(--paper-dark);
    padding: 0.1em 0.35em;
    border-radius: 2px;
}

/* — PARAMETERS TABLE ————— */
.params {
    font-family: 'JetBrains Mono', monospace;
    font-size: 0.85rem;
    display: grid;
    grid-template-columns: auto 1fr;
    gap: 0.3rem 1.5rem;
    padding: 1rem 1.5rem;
    background: var(--paper-dark);
    border-left: 2px solid var(--accent);
    margin: 1rem 0;
}
.params dt { color: var(--ink-soft); }
.params dd { color: var(--ink); font-weight: 500; }

/* — STATUS ————— */
.status {
    font-family: 'JetBrains Mono', monospace;
    font-size: 0.72rem;
    letter-spacing: 0.1em;
    color: var(--ink-soft);
    margin-bottom: 0.8rem;
    padding: 0.4rem 0.8rem;
    background: var(--paper-dark);
    border-left: 2px solid var(--accent);
    display: inline-block;

```

```

    }
    .status.computing { color: var(--signal-red); border-color: var(--
signal-red); }

/* — FOOTER — */
footer {
    margin-top: 3rem;
    padding-top: 1.5rem;
    border-top: 3px double var(--rule);
    font-family: 'JetBrains Mono', monospace;
    font-size: 0.72rem;
    color: var(--ink-soft);
    letter-spacing: 0.08em;
    text-align: center;
    line-height: 2;
}

/* — MOBILE — */
@media (max-width: 640px) {
    body { padding: 1rem 0.8rem; }
    .abstract, .theory, .plot-card { padding: 1rem; }
    .plot-card { padding: 1rem 0.5rem 0.5rem; }
    #plot-container { height: 420px; }
    .byline { gap: 0.8rem; font-size: 0.7rem; }
}
</style>
</head>
<body>
<div class="container">

    <header>
        <div class="eyebrow">
            <span>DTU – Acoustics Laboratory · 1989</span>
            <span>M.Sc. Thesis Archive · §3.4</span>
        </div>
        <h1>
            Sound Propagation in Ducts<br>
            <em>with Absorbent Linings</em>
            <small>Figure 3.10(b) – Modal attenuation for a bulk-reacting
liner</small>
        </h1>
        <div class="byline">
            <span>Michael Howard Grant B.Sc.</span>
            <span>Supervisor: Finn Jacobsen</span>
            <span>web recreation · 2026</span>
        </div>
    </header>

    <div class="abstract">
        <div class="abstract-label">Abstract</div>
        This page re-computes Figure 3.10(b) from the thesis: the axial
attenuation of the (0,0) and (0,2) modes in a rectangular duct lined on
two parallel sides with a bulk-reacting absorber. Scott's transcendental
equation is solved for each frequency using Newton-Raphson for the (0,0)
mode and an arctangent-fixed-point iteration for the (0,2) mode, after
the method of Christie (1971). Lining properties are taken from the
Delany-Bazley formulae as modified by Mechel. The composite least-
attenuation curve is the envelope  $\min\{\text{Re}(\Gamma)\}$  of the two modes. Move the
sliders to explore the parameter space.
    </div>

```

```

<h2 data-num="$">Controls</h2>
<div class="controls">
  <div class="control">
    <label>Flow resistivity  $\sigma$  <span class="value" id="sigma-
val">8500</span> N·s/m4</label>
    <input type="range" id="sigma" min="1000" max="64000" step="500"
value="8500">
  </div>
  <div class="control">
    <label>Duct half-width  $l$  <span class="value" id="l-
val">0.250</span> m</label>
    <input type="range" id="l" min="0.05" max="0.5" step="0.005"
value="0.250">
  </div>
  <div class="control">
    <label>Lining thickness  $d$  <span class="value" id="d-
val">0.050</span> m</label>
    <input type="range" id="d" min="0.005" max="0.3" step="0.005"
value="0.050">
  </div>
  <div class="control">
    <label>Max freq <span class="value" id="fmax-val">4000</span>
Hz</label>
    <input type="range" id="fmax" min="1000" max="8000" step="250"
value="4000">
  </div>
  <button class="btn-reset" id="reset">⏮ Restore Fig.
3.10(b)</button>
</div>

```

```

<h2 data-num="$">Result</h2>
<div id="status" class="status">Computing...</div>

<div class="plot-card">
  <div class="plot-header">
    <span>attenuation · dB per unit duct width</span>
    <span class="fig-id">FIG. 3.10(b) – RECONSTRUCTED</span>
  </div>
  <div id="plot-container"><canvas id="plot"></canvas></div>

  <div class="legend" id="legend">
    <div class="legend-item" data-series="0"><span class="legend-
swatch dashed"></span>(0,0) mode</div>
    <div class="legend-item" data-series="1"><span class="legend-
swatch solid blue"></span>(0,2) mode</div>
    <div class="legend-item" data-series="2"><span class="legend-
swatch solid black"></span>least-attenuated envelope</div>
  </div>

  <div class="plot-caption">
    <strong>Figure 3.10(b).</strong> Produced using the 'Extended'
algorithm (re-implemented in JavaScript).
     $\Gamma$  is the attenuation; the dashed curve is the plane-wave (0,0)
mode, the solid curve the (0,2) mode.
    Current parameters shown below.
  </div>
</div>

<dl class="params" id="params-table">

```

|              |                            |
|--------------|----------------------------|
| $\sigma$     | 8500 N·s/m <sup>4</sup>    |
| $\ell$       | 0.250 m (duct half-width)  |
| $d$          | 0.050 m (lining thickness) |
| $b = d/\ell$ | 0.200                      |
| resolution   | 20 Hz                      |
| $c$          | 344.25 m/s (air, 20 °C)    |
| $\rho$       | 1.205 kg/m <sup>3</sup>    |

## Method

theory

For one mode of propagation the acoustic potentials  $\Phi$  (air) and  $\Theta$  (lining) share a common axial propagation constant  $\Gamma$ . Applying continuity of normal particle velocity and acoustic pressure at the lining-air interface, with a rigid outer wall, leads to Scott's transcendental equation, rewritten by Christie in the dimensionless form:

$$f(w) = w \cdot \tan w + M \cdot \sqrt{(w^2 + p^2)} \cdot \tan(b \cdot \sqrt{(w^2 + p^2)}) = 0$$

where  $w = \sqrt{(\Gamma \ell)^2 + (k \ell)^2}$ ,  $p = \sqrt{(h \ell)^2 - (k \ell)^2}$ ,  $b = d/\ell$ , and  $M = \rho/\rho'$  is the ratio of air density to complex lining density. The lining wave number  $h$  and characteristic impedance  $Z_a$  are obtained from the Delany-Bazley/Mechel formulae with normalised flow resistivity  $A = \sigma/(\rho f)$ .

Once  $w$  is found, the duct propagation constant follows from  $\Gamma = \sqrt{(w/\ell)^2 - k^2}$ , and the attenuation is  $20 \cdot \log_{10}(e) \cdot \text{Re}\{\Gamma\} \approx 8.686 \cdot \text{Re}\{\Gamma\}$  dB per metre. Here it is normalised to the duct half-width  $\ell$ .

The iteration is seeded from the Morse (local-reaction) starting points  $w(0,0) \approx 0$  and  $w(0,2) \approx \pi$ , then continued across the frequency axis using linear extrapolation of the previous solution as the next initial value – exactly as in the 1988 BASIC code of the original thesis.

```
<script>
//
```

---

---

```
// COMPLEX ARITHMETIC - minimal library
// A complex number is {re, im}.
//
```

---

---

```
const C = {
  of: (re, im = 0) => ({ re, im }),
  add: (a, b) => ({ re: a.re + b.re, im: a.im + b.im }),
  sub: (a, b) => ({ re: a.re - b.re, im: a.im - b.im }),
  neg: (a) => ({ re: -a.re, im: -a.im }),
  mul: (a, b) => ({ re: a.re*b.re - a.im*b.im, im: a.re*b.im +
a.im*b.re }),
  scale: (a, s) => ({ re: a.re*s, im: a.im*s }),
  div: (a, b) => {
    const d = b.re*b.re + b.im*b.im;
    return { re: (a.re*b.re + a.im*b.im) / d, im: (a.im*b.re -
a.re*b.im) / d };
  },
  abs: (a) => Math.hypot(a.re, a.im),
  arg: (a) => Math.atan2(a.im, a.re),
  // Principal square root of a complex number
  sqrt: (a) => {
    const r = Math.hypot(a.re, a.im);
    const re = Math.sqrt((r + a.re) / 2);
    const im = Math.sign(a.im || 1) * Math.sqrt((r - a.re) / 2);
    return { re, im };
  },
  // Complex exponential
  exp: (a) => {
    const e = Math.exp(a.re);
    return { re: e * Math.cos(a.im), im: e * Math.sin(a.im) };
  },
  // Complex log
  log: (a) => ({ re: Math.log(Math.hypot(a.re, a.im)), im:
Math.atan2(a.im, a.re) }),
  // sin, cos, tan via complex identities
  sin: (a) => ({
    re: Math.sin(a.re) * Math.cosh(a.im),
    im: Math.cos(a.re) * Math.sinh(a.im)
  }),
  cos: (a) => ({
    re: Math.cos(a.re) * Math.cosh(a.im),
    im: -Math.sin(a.re) * Math.sinh(a.im)
  }),
  tan: (a) => {
    const s = C.sin(a), c = C.cos(a);
    return C.div(s, c);
  },
  pow: (a, s) => {
    // a^s for real s: e^{s·log a}
    const l = C.log(a);
    return C.exp({ re: l.re * s, im: l.im * s });
  }
};
```

```

//
=====
// DELANY-BAZLEY / MECHEL - lining propagation constant and impedance
// Returns { Gamma_a (= propagation constant of lining), Z_a (normalised
to pc) }
// Equations 2.1.1 ( $A \leq 60$ ) and 2.1.2 ( $A \geq 60$ ) in the thesis.
//
=====

function lining(sigma, f, rho, c) {
    const A = sigma / (rho * f);
    const k = 2 * Math.PI * f / c;

    let Gamma_a_over_k, Z_a_over_Z0;
    if (A <= 60) {
        Gamma_a_over_k = C.of(
            0.189 * Math.pow(A, 0.6185),
            1 + 0.0978 * Math.pow(A, 0.6929)
        );
        Z_a_over_Z0 = C.of(
            1 + 0.04885 * Math.pow(A, 0.754),
            -0.087 * Math.pow(A, 0.7307)
        );
    } else {
        //  $\Gamma_a/k = \sqrt{-1.4664 + j \cdot 0.2115 \cdot A}$ 
        Gamma_a_over_k = C.sqrt(C.of(-1.4664, 0.2115 * A));
        //  $Z_a/Z_0 = (A/2\pi + j \cdot 1.4035) / (\Gamma_a/k)$ 
        const num = C.of(A / (2 * Math.PI), 1.4035);
        Z_a_over_Z0 = C.div(num, Gamma_a_over_k);
    }

    return {
        Gamma_a: C.scale(Gamma_a_over_k, k), // actual propagation
constant
        Z_a:      Z_a_over_Z0,                // still normalised to pc
            k
    };
}

//
=====
// SCOTT'S EQUATION
// Given w (complex), and pre-computed M and p2, evaluate f(w) and
f'(w).
//  $f(w) = w \cdot \tan w + M \cdot (w^2 + p^2)^{1/2} \cdot \tan(b \cdot (w^2 + p^2)^{1/2})$ 
//  $f'(w) = (2w + \sin 2w) / (1 + \cos 2w)$ 
//  $+ M \cdot w / \sqrt{(w^2 + p^2)} \cdot [2b \sqrt{(w^2 + p^2)} + \sin(2b \sqrt{(w^2 + p^2)})] / [1 + \cos(2b \sqrt{(w^2 + p^2)})]$ 
// (Equations 3.2.1, 3.2.2 in the thesis.)
//
=====

function f_and_fprime(w, M, p2, b) {
    const w2 = C.mul(w, w);
    const w2p2 = C.add(w2, p2);
    const s = C.sqrt(w2p2); //  $\sqrt{(w^2 + p^2)}$ 
    const bs = C.scale(s, b); //  $b \cdot \sqrt{(w^2 + p^2)}$ 

```

```

// f(w)
const tanW = C.tan(w);
const tanBs = C.tan(bs);
const term1 = C.mul(w, tanW);
const term2 = C.mul(M, C.mul(s, tanBs));
const f = C.add(term1, term2);

// f'(w)
// Part 1: (2w + sin 2w) / (1 + cos 2w) (this equals tan w + w·sec²w)
const twoW = C.scale(w, 2);
const sin2W = C.sin(twoW);
const cos2W = C.cos(twoW);
const part1_num = C.add(twoW, sin2W);
const part1_den = C.add(C.of(1), cos2W);
const part1 = C.div(part1_num, part1_den);

// Part 2: M · (w/s) · [2bs + sin(2bs)] / [1 + cos(2bs)]
const twoBs = C.scale(bs, 2);
const sin2Bs = C.sin(twoBs);
const cos2Bs = C.cos(twoBs);
const part2_num = C.add(twoBs, sin2Bs);
const part2_den = C.add(C.of(1), cos2Bs);
const part2_ratio = C.div(part2_num, part2_den);
const part2 = C.mul(M, C.mul(C.div(w, s), part2_ratio));

const fprime = C.add(part1, part2);

return { f, fprime };
}

//
=====
// NEWTON-RAPHSON iteration for f(w) = 0 (with damping)
// The Newton step is clamped if too large, to avoid leaping to another
// branch.
//
=====
function newtonRaphson(w0, M, p2, b, eps = 1e-11, maxIter = 80) {
  let w = { re: w0.re, im: w0.im };
  for (let i = 0; i < maxIter; i++) {
    const { f, fprime } = f_and_fprime(w, M, p2, b);
    if (C.abs(fprime) < 1e-30) return null;
    const dw = C.div(f, fprime);
    // Damping: clamp |dw| ≤ 0.5 to stay in the current root's basin of
    attraction
    const dwAbs = C.abs(dw);
    const damp = dwAbs > 0.5 ? 0.5 / dwAbs : 1;
    w = C.sub(w, C.scale(dw, damp));
    if (dwAbs < eps) return w;
    if (!isFinite(w.re) || !isFinite(w.im)) return null;
  }
  return w; // best-effort
}

//
=====
// MULLER-STYLE ITERATION (as used in thesis for (0,2) mode)

```

```

// f(w) = w · tan w, g(w) = -M·√(w²+p²) · tan(b·√(w²+p²))
// Fixed-point: w_{n+1} = arctan( g(w_n) / w_{n+1} ) - take principal
branch kn
// using complex arctan: arctan z = (1/2j) · ln((1+jz)/(1-jz))
// Selecting k=1 for (0,2).
//
=====

function complexArctan(z, k = 0) {
  // arctan z = -(j/2) ln((j+z)/(j-z))
  //          = kn + ½(J4 - J2) + (j/2) ln(J1/J3)      (principal branch
form)
  // Here we just use arctan z = (1/(2j)) ln((1+jz)/(1-jz))
  const jz = C.mul(C.of(0, 1), z);
  const num = C.add(C.of(1), jz);
  const den = C.sub(C.of(1), jz);
  const ratio = C.div(num, den);
  const lnR = C.log(ratio); // branch in (-π, π]
  // multiply by 1/(2j) = -j/2
  const res = C.scale(C.mul(C.of(0, -1), lnR), 0.5);
  // Add kn to the real part to select the branch
  return { re: res.re + k * Math.PI, im: res.im };
}

function mullerLike(w0, M, p2, b, branch = 1, eps = 1e-9, maxIter = 200)
{
  let w = w0;
  for (let i = 0; i < maxIter; i++) {
    // g(w) = -M·√(w²+p²) · tan(b·√(w²+p²))
    const s = C.sqrt(C.add(C.mul(w, w), p2));
    const bs = C.scale(s, b);
    const g = C.neg(C.mul(M, C.mul(s, C.tan(bs))));
    // we want w · tan w = g(w), i.e. tan w = g/w, so w = arctan(g/w) on
branch k
    if (C.abs(w) < 1e-30) return null;
    const arg = C.div(g, w);
    const wNext = complexArctan(arg, branch);
    const dw = C.sub(wNext, w);
    w = wNext;
    if (C.abs(dw) < eps) return w;
    if (!isFinite(w.re) || !isFinite(w.im)) return null;
  }
  return w;
}

//
=====

// SOLVE a full frequency sweep for one mode.
// Uses damped Newton-Raphson with linear extrapolation of the previous
two
// solutions as the initial guess - matches the continuation method on
// programme lines 10910-10920 of 'Extended'.
// The 'jump' test on line 10190 rejects a step that is 3× longer than
the
// previous step - a sign the root has migrated to another mode.
// Returns array of { f, att (dB/duct width), ReGamma (1/m) }.
//
=====

function sweep(mode, { sigma, l, d, fmin, fmax, fres, rho, c }) {

```

```

const b = d / l;
const results = [];

// Low-frequency Morse starting values (thesis p.13):
// (0,0) → w near 0 + j·0      (plane-wave mode)
// (0,2) → w near π + j·0      (first higher mode)
let w = (mode === '00') ? C.of(0.01, 0.01) : C.of(Math.PI, 0.01);

// History for linear extrapolation and jump detection
const history = [];

for (let f = fmin; f <= fmax + 1e-6; f += fres) {
  const { Gamma_a, Z_a, k } = lining(sigma, f, rho, c);

  // M = +j·k / (Z_a·Γ_a) — sign taken from the BASIC source line
9230 // CALL Cdiv(0,K,E1,E2,M1,M2) ! jK*rho*C/(Z*Gam)
  // where E1+jE2 = Z_a · Γ_a (already normalised by ρc in
'Material').
  // This gives the physically correct mode behaviour; the typeset
text on
  // p.12 of the thesis has a sign error.
  const M = C.div(C.of(0, +k), C.mul(Z_a, Gamma_a));

  // p2 = (hℓ)2 - (kℓ)2 = -(Γ_a·ℓ)2 - (kℓ)2 since h = -j·Γ_a
  const Gal = C.scale(Gamma_a, 1);
  const kl = k * l;
  const p2 = C.sub(C.neg(C.mul(Gal, Gal)), C.of(kl * kl));

  // Initial guess: linear extrapolation of the last two w values
  let wInit = w;
  if (history.length >= 2) {
    const a = history[history.length - 2];
    const b_ = history[history.length - 1];
    wInit = { re: 2 * b_.re - a.re, im: 2 * b_.im - a.im };
  }

  const wSol = newtonRaphson(wInit, M, p2, b, 1e-11, 80);
  if (!wSol) {
    results.push({ f, att: null, ReGamma: null });
    continue;
  }

  // Jump detection (thesis programme line 10190)
  let jumped = false;
  if (history.length >= 2) {
    const prev = history[history.length - 1];
    const prev2 = history[history.length - 2];
    const dNew = Math.hypot(wSol.re - prev.re, wSol.im - prev.im);
    const dPrev = Math.hypot(prev.re - prev2.re, prev.im - prev2.im);
    if (dNew > 3 * Math.max(dPrev, 0.05)) jumped = true;
  }
  if (jumped) {
    results.push({ f, att: null, ReGamma: null });
    continue;
  }

  // Extract Γ from w: Γ2 = (w/ℓ)2 - k2
  const w2 = C.mul(wSol, wSol);

```

```

    const Gamma2 = C.scale(C.sub(w2, C.of(k1 * k1)), 1 / (l * l));
    const Gamma = C.sqrt(Gamma2);

    const ReGamma = Math.abs(Gamma.re);
    const att_dB_per_2l = 8.6858896 * ReGamma * (2 * l);    // dB per
duct width 2l

    results.push({ f, att: att_dB_per_2l, ReGamma });

    // Update state
    w = wSol;
    history.push(wSol);
    if (history.length > 6) history.shift();
  }

  return results;
}

//
=====
// CHART SETUP
//
=====
const ctx = document.getElementById('plot').getContext('2d');

const CHART_COLORS = {
  red:    '#b23a2d',
  blue:   '#1e4b7c',
  black:  '#1a1815',
  grid:   'rgba(42, 38, 32, 0.12)',
  gridMajor: 'rgba(42, 38, 32, 0.25)',
  text:   '#4a4640'
};

Chart.defaults.font.family = "'JetBrains Mono', monospace";
Chart.defaults.font.size = 11;
Chart.defaults.color = CHART_COLORS.text;

const chart = new Chart(ctx, {
  type: 'line',
  data: {
    datasets: [
      {
        label: '(0,0) mode',
        data: [],
        borderColor: CHART_COLORS.red,
        backgroundColor: CHART_COLORS.red,
        borderDash: [5, 4],
        borderWidth: 1.8,
        pointRadius: 0,
        tension: 0.1,
        spanGaps: false
      },
      {
        label: '(0,2) mode',
        data: [],
        borderColor: CHART_COLORS.blue,

```

```

        backgroundColor: CHART_COLORS.blue,
        borderWidth: 1.8,
        pointRadius: 0,
        tension: 0.1,
        spanGaps: false
    },
    {
        label: 'envelope min Re( $\Gamma$ )',
        data: [],
        borderColor: CHART_COLORS.black,
        backgroundColor: CHART_COLORS.black,
        borderWidth: 2.5,
        pointRadius: 0,
        tension: 0.05,
        spanGaps: false
    }
]
},
options: {
    responsive: true,
    maintainAspectRatio: false,
    animation: { duration: 200 },
    interaction: { mode: 'nearest', axis: 'x', intersect: false },
    plugins: {
        legend: { display: false },
        tooltip: {
            backgroundColor: '#1a1815',
            titleFont: { family: "'JetBrains Mono', monospace", size: 11 },
            bodyFont: { family: "'JetBrains Mono', monospace", size: 11 },
            titleColor: '#f4efe4',
            bodyColor: '#f4efe4',
            borderColor: '#8a6c3a',
            borderWidth: 1,
            padding: 10,
            cornerRadius: 0,
            displayColors: true,
            callbacks: {
                title: (items) => `f = ${items[0].parsed.x.toFixed(1)} Hz`,
                label: (item) => `${item.dataset.label}:
${item.parsed.y.toFixed(2)} dB/width`
            }
        }
    },
    scales: {
        x: {
            type: 'logarithmic',
            title: {
                display: true,
                text: 'frequency / Hz',
                font: { family: "'EB Garamond', serif", size: 13, style:
'italic' },
                color: CHART_COLORS.text
            },
            grid: { color: CHART_COLORS.grid, drawTicks: true, tickColor:
CHART_COLORS.text },
            border: { color: CHART_COLORS.black, width: 1.5 },
            ticks: {
                color: CHART_COLORS.text,
                callback: function(val) {
                    const candidates = [62.5, 125, 250, 500, 1000, 2000, 4000,
8000];

```

```

        for (const c of candidates) if (Math.abs(val - c) / c <
0.02) return c.toString();
        return '';
    }
    },
    y: {
        beginAtZero: true,
        min: 0,
        max: 10,
        title: {
            display: true,
            text: 'attenuation / dB per duct width (2l)',
            font: { family: "'EB Garamond', serif", size: 13, style:
'italic' },
            color: CHART_COLORS.text
        },
        grid: { color: CHART_COLORS.grid, drawTicks: true, tickColor:
CHART_COLORS.text },
        border: { color: CHART_COLORS.black, width: 1.5 },
        ticks: { color: CHART_COLORS.text, stepSize: 1 }
    }
}
});

//
=====
// RUN the computation and update the chart
//
=====

const statusEl = document.getElementById('status');

function recompute() {
    const sigma = parseFloat(document.getElementById('sigma').value);
    const l      = parseFloat(document.getElementById('l').value);
    const d      = parseFloat(document.getElementById('d').value);
    const fmax   = parseFloat(document.getElementById('fmax').value);

    // UI labels
    document.getElementById('sigma-val').textContent = sigma.toFixed(0);
    document.getElementById('l-val').textContent     = l.toFixed(3);
    document.getElementById('d-val').textContent     = d.toFixed(3);
    document.getElementById('fmax-val').textContent = fmax.toFixed(0);
    document.getElementById('p-sigma').textContent  = sigma.toFixed(0) +
' N·s/m4';
    document.getElementById('p-l').textContent       = l.toFixed(3) + ' m
(duct half-width)';
    document.getElementById('p-d').textContent       = d.toFixed(3) + ' m
(lining thickness)';
    document.getElementById('p-b').textContent       = (d / l).toFixed(3);

    // Choose a sensible resolution: 20 Hz for the reference case, fewer
steps otherwise
    const fmin = 62.5;
    const fres = Math.max(5, Math.min(25, Math.round(fmax / 180)));
    document.getElementById('p-res').textContent = fres + ' Hz';

```

```

statusEl.textContent = '● Solving Scott\'s equation for both modes...';
statusEl.classList.add('computing');

setTimeout(() => { // let the UI repaint
  const params = { sigma, l, d, fmin, fmax, fres, rho: 1.205, c:
344.25 };

  const t0 = performance.now();
  const res00 = sweep('00', params);
  const res02 = sweep('02', params);
  const dt = performance.now() - t0;

  // Composite: envelope = min of the two Re( $\Gamma$ ) at each frequency
  const envelope = res00.map((r00, i) => {
    const r02 = res02[i];
    if (r00.att == null && r02.att == null) return { f: r00.f, att:
null };
    if (r00.att == null) return { f: r00.f, att: r02.att };
    if (r02.att == null) return { f: r00.f, att: r00.att };
    return { f: r00.f, att: Math.min(r00.att, r02.att) };
  });

  // Filter out any unphysical spikes (very rarely happens near cut-
on)
  const clean = (arr) => arr.map(r => ({
    x: r.f,
    y: (r.att == null || !isFinite(r.att) || r.att < 0 || r.att > 100)
? null : r.att
  }));

  chart.data.datasets[0].data = clean(res00);
  chart.data.datasets[1].data = clean(res02);
  chart.data.datasets[2].data = clean(envelope);
  chart.update('none');

  statusEl.textContent = `● ${res00.length} frequency points ·
computed in ${dt.toFixed(0)} ms`;
  statusEl.classList.remove('computing');
}, 20);
}

//
=====
// LEGEND: click to toggle series
//
=====
document.querySelectorAll('.legend-item').forEach(el => {
  el.addEventListener('click', () => {
    const idx = +el.dataset.series;
    const meta = chart.getDatasetMeta(idx);
    meta.hidden = !meta.hidden;
    el.classList.toggle('disabled', meta.hidden);
    chart.update();
  });
});

```

```

//
=====
// Wire up controls (debounced)
//
=====
let debounceTimer;
const debouncedRecompute = () => {
  clearTimeout(debounceTimer);
  debounceTimer = setTimeout(recompute, 120);
};
['sigma', 'l', 'd', 'fmax'].forEach(id => {
  document.getElementById(id).addEventListener('input',
debouncedRecompute);
});

document.getElementById('reset').addEventListener('click', () => {
  document.getElementById('sigma').value = 8500;
  document.getElementById('l').value      = 0.25;    // half of 0.5 m
duct width
  document.getElementById('d').value      = 0.05;
  document.getElementById('fmax').value   = 4000;
  recompute();
});

// Kick off
recompute();
</script>

</body>
</html>

```

— end of listing —

## References

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